# Integration by Substitution and Using Partial Fractions

# 13.5

## Introduction

The first technique described here involves making a substitution to simplify an integral. We let a new variable equal a complicated part of the function we are trying to integrate. Choosing the correct substitution often requires experience. This skill develops with practice.

Often the technique of partial fractions can be used to write an algebraic fraction as the sum of simpler fractions. On occasions this means that we can then integrate a complicated algebraic fraction. We shall explore this approach in the second half of the section.

## Prerequisites

Before starting this Section you should ...

- be able to find a number of simple definite and indefinite integrals
- be able to use a table of integrals
- be familiar with the technique of expressing an algebraic fraction as the sum of its partial fractions

## Learning Outcomes

On completion you should be able to ...

- make simple substitutions in order to find definite and indefinite integrals
- understand the technique used for evaluating integrals of the form $\displaystyle\int \frac{f'(x)}{f(x)}dx$
- use partial fractions to express an algebraic fraction in a simpler form and integrate it

# 1. Making a substitution

The technique described here involves making a substitution in order to simplify an integral. We let a new variable, $u$ say, equal a more complicated part of the function we are trying to integrate. The choice of which substitution to make often relies upon experience: don't worry if at first you cannot see an appropriate substitution. This skill develops with practice. However, it is not simply a matter of changing the variable - care must be taken with the differential form $dx$ as we shall see. The technique is illustrated in the following Example.

## Example 19

Find $\int (3x+5)^6 dx$.

**Solution**

First look at the function we are trying to integrate: $(3x+5)^6$. It looks quite complicated to integrate. Suppose we introduce a new variable, $u$, such that $u = 3x + 5$. Doing this means that the function we must integrate becomes $u^6$. Would you not agree that this looks a much simpler function to integrate than $(3x+5)^6$? There is a slight complication however. The new function of $u$ must be integrated with respect to $u$ and not with respect to $x$. This means that we must take care of the term $dx$ correctly.

**Long Method** $\quad u = 3x + 5$ so $\dfrac{du}{dx} = 3,$ or $\dfrac{dx}{du} = \dfrac{1}{3}$

Let
$$\begin{aligned} I = \int (3x+5)^6\, dx &= \int u^6\, dx && \text{(substituting for } 3x+5) \\ &= \int u^6 \frac{dx}{du}\, du && \text{(to change from } x \text{ to } u) \\ &= \int u^6 \frac{1}{3}.\, du && \text{(substituting for } \tfrac{dx}{du}) \\ &= \frac{1}{3}\int u^6\, dx = \frac{u^7}{21} + \text{constant} \end{aligned}$$

**Short Method** $\quad u = 3x + 5$ so $\dfrac{du}{dx} = 3,$ so $dx = \dfrac{1}{3}du$

Let $\quad I = \displaystyle\int (3x+5)^6\, dx = \int u^6\, dx = \int u^6.\frac{1}{3}.\, du = \frac{1}{3}\int u^6\, du = \frac{u^7}{21} + \text{constant}$

To finish off we must rewrite this answer in terms of the original variable $x$ and replace $u$ by $3x+5$:

$$\int (3x+5)^6 dx = \frac{(3x+5)^7}{21} + c$$

In practice the short method is generally used but mathematicians don't like to separate the '$dx$' from the '$du$' as in the statement '$dx = \frac{1}{3}du$' as it is meaningless mathematically (but it works!). In the future we will use the short method, with apologies to the mathematicians!

**Task**

By making the substitution $u = \sin x$ find $\displaystyle\int \cos x \sin^2 x\, dx$

You are given the substitution $u = \sin x$. Find $\dfrac{du}{dx}$:

| **Your solution** |
|---|
| |

| **Answer** |
|---|
| $\dfrac{du}{dx} = \cos x$ |

Now make the substitution, simplify the result, and finally perform the integration:

| **Your solution** |
|---|
| |

| **Answer** |
|---|
| $\displaystyle\int \cos x \sin^2 x\, dx$ simplifies to $\displaystyle\int u^2 du$. The final answer is $\dfrac{1}{3}\sin^3 x + c$. |

## Exercise

Use suitable substitutions to find

(a) $\displaystyle\int (4x+1)^7 dx$ (b) $\displaystyle\int t^2 \sin(t^3+1) dt$ (Hint: you need to simplify $\sin(t^3+1)$)

| **Answer** |
|---|
| (a) $\dfrac{(4x+1)^8}{32} + c$ (b) $-\dfrac{\cos(t^3+1)}{3} + c$ |

## 2. Substitution and definite integration

If you are dealing with definite integrals (ones with limits of integration) you must be particularly careful when you substitute. Consider the following example.

### Example 20

Find the definite integral $\displaystyle\int_2^3 t\sin(t^2)dt$ by making the substitution $u = t^2$.

**Solution**

Note that if $u = t^2$ then $\dfrac{du}{dt} = 2t$ so that $dt = \dfrac{du}{2t}$. We find

$$\int_{t=2}^{t=3} t\sin(t^2)dt = \int_{t=2}^{t=3} t\sin u\,\frac{du}{2t} = \frac{1}{2}\int_{t=2}^{t=3} \sin u\,du$$

An important point to note is that the limits of integration are limits on the variable $t$, not $u$. To emphasise this they have been written explicitly as $t = 2$ and $t = 3$. When we integrate with respect to the variable $u$, the limits must be written in terms of $u$. From the substitution $u = t^2$, note that when $t = 2$ then $u = 4$ and when $t = 3$ then $u = 9$ so the integral becomes

$$\frac{1}{2}\int_{u=4}^{u=9} \sin u\,du = \frac{1}{2}\Big[-\cos u\Big]_4^9 = \frac{1}{2}(-\cos 9 + \cos 4) = 0.129 \quad \text{to 3 d.p.}$$

### Exercise

Use suitable substitutions to find (a) $\displaystyle\int_1^2 (2x+3)^7 dx$, (b) $\displaystyle\int_0^1 3t^2 e^{t^3} dt$.

**Answer**

(a) $u = 2x + 3$ is suitable; $3.359 \times 10^5$ to 4 sig. figs. (b) 1.718 to 3 d.p.

## 3. Integrals giving rise to logarithms

**Example 21**

Find $\displaystyle\int \frac{3x^2+1}{x^3+x+2}dx$

**Solution**

Let us consider what happens when we make the substitution $z = x^3 + x + 2$. Note that

$$\frac{dz}{dx} = 3x^2 + 1 \qquad \text{so that we can write} \qquad dz = (3x^2+1)dx$$

Then

$$\int \frac{3x^2+1}{x^3+x+2}dx = \int \frac{1}{z}dz = \ln|z| + c = \ln|x^3+x+2|$$

Note that in the last Example, the numerator of the integrand $(3x^2 + 1)$ is the derivative of the denominator $(x^3 + x + 2)$. The result is the logarithm of the denominator. This is a special case of the following rule:

**Key Point 7**

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Note that it is the modulus of $f(x)$ in the answer.

**Task**

Write down, purely by inspection, the following integrals:

(a) $\displaystyle\int \frac{1}{x+1}dx$, (b) $\displaystyle\int \frac{2x}{x^2+8}dx$, (c) $\displaystyle\int \frac{1}{x-3}dx$.

Hint: In each case the numerator of the integrand is the derivative of the denominator.

**Your solution**

(a) (b) (c)

**Answer**

(a) $\ln|x+1|+c$, (b) $\ln|x^2+8|+c$, (c) $\ln|x-3|+c$

**Task**

Evaluate the definite integral $\displaystyle\int_2^4 \frac{3t^2+2t}{t^3+t^2+1}dt$.

**Your solution**

**Answer**

$$\Big[\ \ln|t^3+t^2+1|\ \Big]_2^4 = \ln 81 - \ln 13 = 1.83$$

Sometimes it is necessary to make slight adjustments to the integrand to obtain a form for which the rule in Key Point 7 is suitable. Consider the next Example.

## Example 22

Find the indefinite integral $\displaystyle\int \frac{x^2}{x^3+1}dx$.

**Solution**

In this Example the derivative of the denominator is $3x^2$ whereas the numerator is just $x^2$. We adjust the numerator as follows:

$$\int \frac{x^2}{x^3+1}dx = \frac{1}{3}\int \frac{3x^2}{x^3+1}dx \quad \text{and integrate by the rule to get} \quad \tfrac{1}{3}\ln|x^3+1|+c$$

Note that the sort of procedure in the last Example is only possible because we can move **constant factors** through the integral sign. It would be wrong to try to move terms involving the **variable** $x$ in a similar way.

## Exercise

Write down the result of finding the following integrals.

(a) $\displaystyle\int \frac{1}{x}dx$, (b) $\displaystyle\int \frac{2t}{t^2+1}dt$, (c) $\displaystyle\int \frac{1}{2x+5}dx$, (d) $\displaystyle\int \frac{2}{3x-2}dx$.

**Answer**

(a) $\ln|x|+c$, (b) $\ln|t^2+1|+c$, (c) $\frac{1}{2}\ln|2x+5|+c$, (d) $\frac{2}{3}\ln|3x-2|+c$.

## 4. Integration using partial fractions

Sometimes expressions which at first sight look impossible to integrate using the techniques already met may in fact be integrated by first expressing them as simpler partial fractions, and then using the techniques described earlier in this Section. Consider the following Task.

**Task**

Express $\dfrac{23-x}{(x-5)(x+4)}$ as the sum of its partial fractions.

Hence find $\displaystyle\int \frac{23-x}{(x-5)(x+4)}dx$

First produce the partial fractions. Write the fraction in the form $\dfrac{A}{x-5}+\dfrac{B}{x+4}$ and find $A, B$.

**Your solution**

**Answer**

$A = 2,\ B = -3$

Now integrate each term separately:

**Your solution**

$$\int \frac{23-x}{(x-5)(x+4)}dx = \int \frac{A}{x-5}dx + \int \frac{B}{x+4}dx =$$

**Answer**

$2\ln|x-5| - 3\ln|x+4| + c$

## Exercises

By expressing the following in partial fractions, evaluate each integral:

1. $\displaystyle\int \frac{1}{x^3+x}dx$

2. $\displaystyle\int \frac{13x-4}{6x^2-x-2}dx$

3. $\displaystyle\int \frac{1}{(x+1)(x-5)}dx$

4. $\displaystyle\int \frac{2x}{(x-1)^2(x+1)}dx$

**Answers**

1. $\ln|x| - \dfrac{1}{2}\ln|x^2+1| + c$

2. $\dfrac{3}{2}\ln|2x+1| + \dfrac{2}{3}\ln|3x-2| + c$

3. $\dfrac{1}{6}\ln|x-5| - \dfrac{1}{6}\ln|x+1| + c$

4. $-\dfrac{1}{2}\ln|x+1| + \dfrac{1}{2}\ln|x-1| - \dfrac{1}{x-1} + c$