The Area Bounded by a Curve

13.3

Introduction

One of the important applications of integration is to find the area bounded by a curve. Often such an area can have a physical significance like the work done by a motor, or the distance travelled by a vehicle. In this Section we explain how such an area is calculated.

Prerequisites

Before starting this Section you should . . .

- understand integration as the reverse of differentiation
- be able to use a table of integrals
- be able to evaluate definite integrals
- be able to sketch graphs of common functions including polynomials, simple rational functions, exponential functions and trigonometric functions

Learning Outcomes

On completion you should be able to . . .

- find the area bounded by a curve and the x-axis
- find the area between two curves

1. Calculating the area under a curve

Let us denote the area under $y = f(x)$ between a fixed point a and a variable point x by $A(x)$:

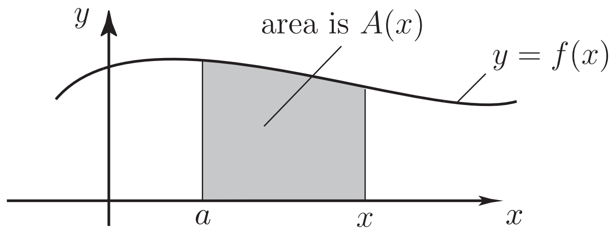


Figure 6

$A(x)$ is clearly a function of x since as the upper limit changes so does the area. How does the area change if we change the upper limit by a very small amount δx? See Figure 7 below.

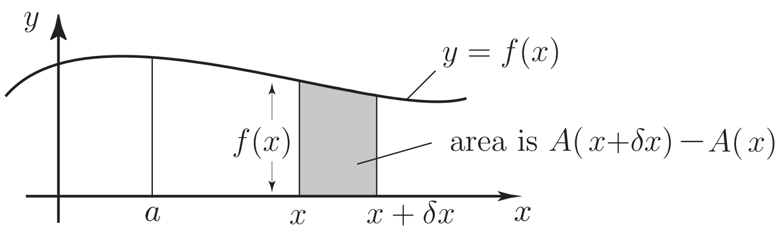


Figure 7

To a good approximation the change in the area is:

$$A(x + \delta x) - A(x) \approx f(x)\delta x$$

[This is because the shaded area is approximately a rectangle with base δx and height $f(x)$.] This approximation gets better and better as δx gets smaller and smaller. Rearranging gives:

$$f(x) \approx \frac{A(x + \delta x) - A(x)}{\delta x}$$

Clearly, in the limit as $\delta x \to 0$ we have

$$f(x) = \lim_{\delta x \to 0} \frac{A(x + \delta x) - A(x)}{\delta x}$$

But this limit on the right-hand side is the **derivative** of $A(x)$ with respect to x so

$$f(x) = \frac{dA(x)}{dx}$$

Thus $A(x)$ is an **indefinite integral** of $f(x)$ and we can therefore write:

$$A(x) = \int f(x)dx$$

Now the area under the curve from a to b is clearly $A(b) - A(a)$. But remembering our shorthand notation for this difference, introduced in the last Section we have, finally

$$A(b) - A(a) \equiv \Big[A(x) \Big]_a^b = \int_a^b f(x)dx$$

We conclude that the area under the curve $y = f(x)$ from a to b is given by the definite integral of $f(x)$ from a to b.

2. The area bounded by a curve lying above the x-axis

Consider the graph of the function $y = f(x)$ shown in Figure 8. Suppose we are interested in calculating the area underneath the graph and above the x-axis, between the points where $x = a$ and $x = b$. When such an area lies entirely above the x-axis, as is clearly the case here, this area is given by the definite integral $\int_a^b f(x)\,dx$.

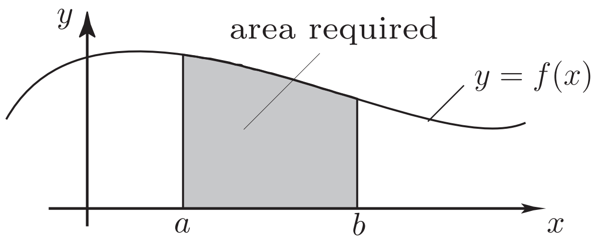


Figure 8

Key Point 4

The area under the curve $y = f(x)$, between $x = a$ and $x = b$ is given by $\displaystyle\int_a^b f(x)\,dx$

when the curve lies entirely above the x-axis between a and b.

Example 12

Calculate the area bounded $y = x^{-1}$ and the x-axis, between $x = 1$ and $x = 4$.

Solution

Below is a graph of $y = x^{-1}$. The area required is shaded; it lies entirely above the x-axis.

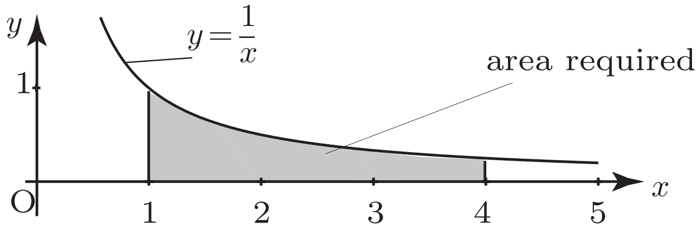
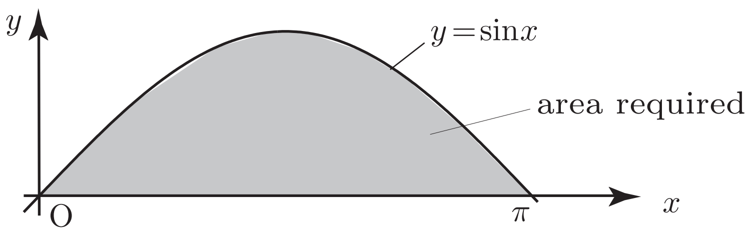


Figure 9

$$\text{area} \quad = \int_1^4 \frac{1}{x}dx = \Big[\ln|x|\Big]_1^4 \quad = \ln 4 - \ln 1 \ = \ln 4 = 1.386 \text{ (3 d.p.)}$$

Task

Find the area bounded by the curve $y = \sin x$ and the x-axis between $x = 0$ and $x = \pi$. (The required area is shown in the figure. Note that it lies entirely above the x-axis.)



Your solution

Answer

$\displaystyle\int_0^{\pi} \sin x \, dx = \Big[-\cos x \Big]_0^{\pi} = 2.$

Task

Find the area under $f(x) = \mathrm{e}^{2x}$ from $x = 1$ to $x = 3$ given that the exponential function e^{2x} is always positive.

Your solution

Answer

area $= \displaystyle\int_1^3 \mathrm{e}^{2x} dx = \left[\frac{1}{2}\mathrm{e}^{2x} \right]_1^3 = 198$ to 3 significant figures.

Example 13

The figure shows the graphs of $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \frac{1}{2}\pi$. The two graphs intersect at the point where $x = \frac{1}{4}\pi$. Find the shaded area.

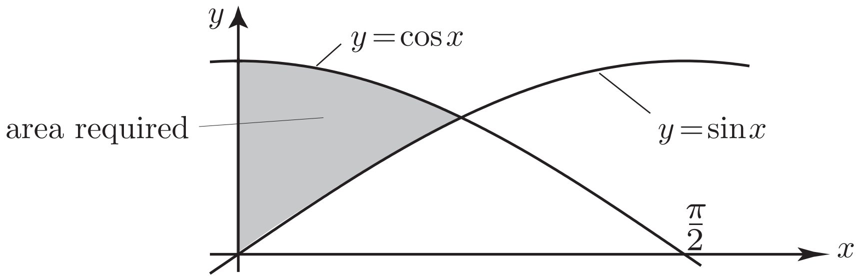


Figure 10

Solution

To find the shaded area we could calculate the area under the graph of $y = \sin x$ for x between 0 and $\frac{1}{4}\pi$, and subtract this from the area under the graph of $y = \cos x$ between the same limits. Alternatively the two processes can be combined into one and we can write

$$\begin{aligned}
\text{shaded area} &= \int_0^{\pi/4} (\cos x - \sin x)dx \\
&= \Big[\sin x + \cos x \Big]_0^{\pi/4} \\
&= \left(\sin \tfrac{1}{4}\pi + \cos \tfrac{1}{4}\pi\right) - (\sin 0 + \cos 0) \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1) = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1
\end{aligned}$$

So the numeric value of the integral is $\frac{2}{\sqrt{2}} - 1 = 0.414$ to 3 d.p.. (Alternatively you can use your calculator to obtain this result directly by evaluating $\sin \dfrac{\pi}{4}$ and $\cos \dfrac{\pi}{4}$.)

Exercises

In each question you should check that the required area lies entirely above the horizontal axis.

1. Find the area under the curve $y = 7x^2$ and above the x-axis between $x = 2$ and $x = 5$.
2. Find the area bounded by the curve $y = x^3$ and the x-axis between $x = 0$ and $x = 2$.
3. Find the area bounded by the curve $y = 3t^2$ and the t-axis between $t = -3$ and $t = 3$.
4. Find the area under $y = x^{-2}$ between $x = 1$ and $x = 10$.

Answer

1. 273, 2. 4, 3. 54, 4. 0.9.

3. The area bounded by a curve, not entirely above the x-axis

Figure 11 shows a graph of $y = -x^2 + 1$.

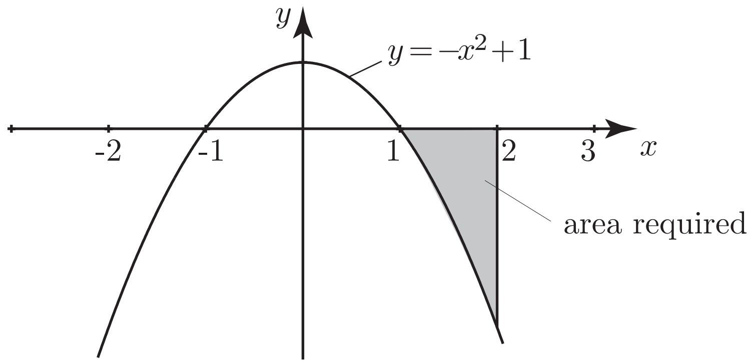


Figure 11

The shaded area is bounded by the x-axis and the curve, but lies entirely below the x-axis. Let us evaluate the integral $\int_1^2 (-x^2 + 1)dx$.

$$\begin{aligned}\int_1^2 (-x^2+1)dx &= \left[-\frac{x^3}{3} + x\right]_1^2 \\ &= \left(-\frac{2^3}{3} + 2\right) - \left(-\frac{1^3}{3} + 1\right) \\ &= -\frac{7}{3} + 1 = -\frac{4}{3}\end{aligned}$$

The evaluation of the area yields a negative quantity. There is, of course, no such thing as a negative area. The area is actually $\frac{4}{3}$, and the negative sign is an indication that the area lies below the x-axis. (However, in applications of integration such as work/energy or distance travelled in a given direction negative values can be meaningful.)

If an area contains parts both above and below the horizontal axis, care must be taken when calculating this area. It is necessary to determine which parts of the graph lie above the horizontal axis and which lie below. Separate integrals need to be calculated for each 'piece' of the graph. This idea is illustrated in the next Example.

Example 14

Find the total area enclosed by the curve $y = x^3 - 5x^2 + 4x$ and the x-axis between $x = 0$ and $x = 3$.

Solution

We need to determine which parts of the graph lie above and which lie below the x-axis. To do this it is helpful to consider where the graph cuts the x-axis. So we consider the function $x^3 - 5x^2 + 4x$ and look for its zeros

$$x^3 - 5x^2 + 4x = x(x^2 - 5x + 4) = x(x-1)(x-4)$$

So the graph cuts the x-axis when $x = 0$, $x = 1$ and $x = 4$. Also, when x is large and positive, y is large and positive since the term involving x^3 dominates. When x is large and negative, y is large and negative for the same reason. With this information we can sketch a graph showing the required area:

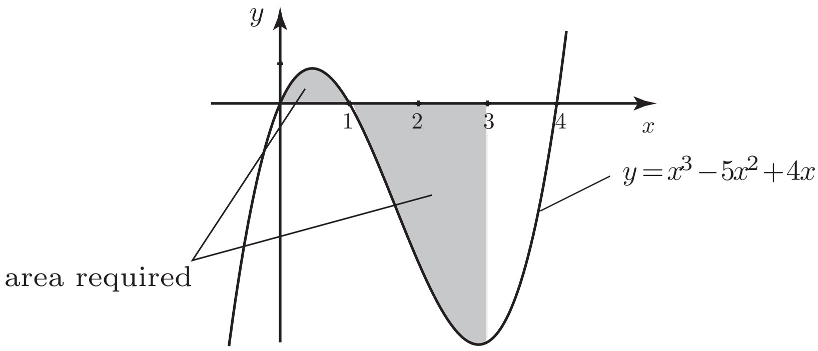


Figure 12

From the graph we see that the required area lies partly above the x-axis (when $0 \leq x \leq 1$) and partly below (when $1 \leq x \leq 3$). So we evaluate the integral in two parts: Firstly:

$$\int_0^1 (x^3 - 5x^2 + 4x)dx = \left[\frac{x^4}{4} - \frac{5x^3}{3} + \frac{4x^2}{2}\right]_0^1 = \left(\frac{1}{4} - \frac{5}{3} + 2\right) - (0) = \frac{7}{12}$$

This is the part of the required area which lies above the x-axis. Secondly:

$$\begin{aligned}\int_1^3 (x^3 - 5x^2 + 4x)dx &= \left[\frac{x^4}{4} - \frac{5x^3}{3} + \frac{4x^2}{2}\right]_1^3 \\ &= \left(\frac{81}{4} - \frac{135}{3} + 18\right) - \left(\frac{1}{4} - \frac{5}{3} + 2\right) = -\frac{22}{3}\end{aligned}$$

This represents the part of the required area which lies below the x-axis. The actual area is $\frac{22}{3}$. Combining the results of the two separate calculations we can find the total area bounded by the curve:

$$\text{area } = \frac{7}{12} + \frac{22}{3} = \frac{95}{12}$$

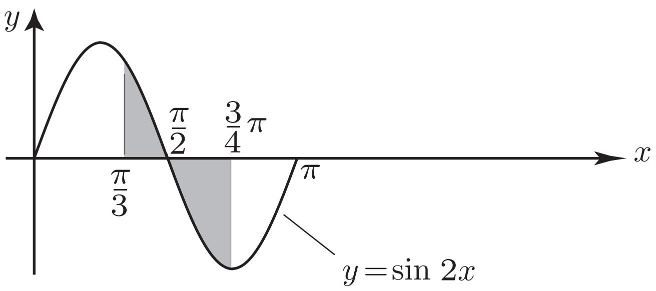
Task

(a) Sketch the graph of $y = \sin 2x$ for $0 \leq x \leq \pi$.
(b) Find the total area bounded by the curve and the x-axis between $x = \frac{1}{3}\pi$ and $x = \frac{3}{4}\pi$.

(a) Sketch the graph and indicate the required area noting where the graph crosses the x-axis:

Your solution

Answer



(b) Perform the integration in two parts to obtain the required area:

Your solution

Answer

$\displaystyle\int_{\pi/3}^{\pi/2} \sin 2x\, dx = \frac{1}{4}$ and $\displaystyle\int_{\pi/2}^{3\pi/4} \sin 2x dx = -\frac{1}{2}$.

The required area is $\dfrac{1}{4} + \dfrac{1}{2} = \dfrac{3}{4}$.

Exercises

1. Find the total area enclosed between the x-axis and the curve $y = x^3$ between $x = -1$ and $x = 1$.
2. Find the area under $y = \cos 2t$ from $t = 0$ to $t = 0.5$.
3. Find the area enclosed by $y = 4 - x^2$ and the x axis

   (a) from $x = 0$ to $x = 2$, (b) from $x = -2$ to $x = 1$, (c) from $x = 1$ to $x = 3$.
4. Calculate the area enclosed by the curve $y = x^3$ and the line $y = x$.
5. Find the area bounded by $y = \mathrm{e}^x$, the y-axis and the line $x = 2$.
6. Find the area enclosed between $y = x(x-1)(x-2)$ and the x axis.

Answers

1. 0.5 2. 0.4207 3. (a) $\frac{16}{3}$, (b) 9, (c) 4 4. 0.5 5. $\mathrm{e}^2 - 1$ 6. $\frac{1}{2}$