Definite Integrals

13.2

Introduction

When you were first introduced to integration as the reverse of differentiation, the integrals you dealt with were **indefinite integrals**. The result of finding an indefinite integral is usually a function plus a constant of integration. In this Section we introduce **definite integrals**, so called because the result will be a definite answer, usually a number, with no constant of integration. Definite integrals have many applications, for example in finding areas bounded by curves, and finding volumes of solids.

Prerequisites

Before starting this Section you should ...

- understand integration as the reverse of differentiation
- be able to use a table of integrals

Learning Outcomes

On completion you should be able to ...

- find simple definite integrals
- handle some integrals involving an infinite limit of integration

1. Definite integrals

We saw in the previous Section that $\int f(x)\,dx = F(x) + c$ where $F(x)$ is that function which, when differentiated, gives $f(x)$. That is, $\dfrac{dF}{dx} = f(x)$. For example,

$$\int \sin(3x)\,dx = -\frac{\cos(3x)}{3} + c$$

Here, $f(x) = \sin(3x)$ and $F(x) = -\frac{1}{3}\cos(3x)$ We now consider a definite integral which is simply an indefinite integral but with numbers written to the upper and lower right of the integral sign. The quantity

$$\int_a^b f(x)\,dx$$

is called the definite integral of $f(x)$ from a to b. The numbers a and b are known as the **lower limit** and **upper limit** respectively of the integral. We *define*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

so that a definite integral is usually a number. The *meaning* of a definite integral will be developed in later Sections. For the present we concentrate on the *process* of evaluating definite integrals.

2. Evaluating definite integrals

When you evaluate a definite integral the result will usually be a number. To see how to evaluate a definite integral consider the following Example.

Example 9

Find the definite integral of x^2 from 1 to 4; that is, find $\int_1^4 x^2\,dx$

Solution

$$\int x^2\,dx = \tfrac{1}{3}x^3 + c$$

Here $f(x) = x^2$ and $F(x) = \frac{x^3}{3}$. Thus, according to our definition

$$\int_1^4 x^2\,dx = F(4) - F(1) = \frac{4^3}{3} - \frac{1^3}{3} = 21$$

Writing $F(b) - F(a)$ each time we calculate a definite integral becomes laborious so we replace this difference by the shorthand notation $\Big[\, F(x) \Big]_a^b$. Thus

$$\Big[\, F(x) \Big]_a^b \equiv F(b) - F(a)$$

Thus, from now on, we shall write

$$\int_a^b f(x)\,dx = \Big[\, F(x) \Big]_a^b$$

so that, for example

$$\int_1^4 x^2\,dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = 21$$

Example 10

Find the definite integral of $\cos x$ from 0 to $\frac{\pi}{2}$; that is, find $\displaystyle\int_0^{\pi/2} \cos x\,dx$.

Solution

Since $\int \cos x\,dx = \sin x + c$ then

$$\begin{aligned} \int_0^{\pi/2} \cos x\,dx &= \Big[\, \sin x \Big]_0^{\pi/2} \\ &= \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1 - 0 = 1 \end{aligned}$$

Always remember, that if you use a calculator to evaluate any trigonometric functions, you must work in **radian mode**.

Task

Find the definite integral of $x^2 + 1$ from 1 to 2; that is; find $\displaystyle\int_1^2 (x^2+1)\,dx$

First perform the integration:

Your solution

Answer

$\left[\frac{1}{3}x^3 + x\right]_1^2$.

Now insert the limits of integration, the upper limit first, and hence evaluat the integral:

Your solution

Answer

$\left(\dfrac{8}{3}+2\right)-\left(\dfrac{1}{3}+1\right)=\dfrac{10}{3}$ or 3.333 (3 d.p.).

Task

Find $\displaystyle\int_2^1 (x^2+1)\,dx$.

This Task is very similar to the previous Task. Note the limits have been interchanged:

Your solution

Answer

$\left[\dfrac{1}{3}x^3+x\right]_2^1=\left[\dfrac{1}{3}+1\right]-\left[\dfrac{8}{3}+2\right]=-\dfrac{10}{3}.$

Note from these two Tasks that interchanging the limits of integration, changes the sign of the answer.

Key Point 3

If you interchange the limits, you must change the sign:

$$\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$$

When a spring is fixed at one end and stretched at the free end it exerts a restoring force that is proportional to the displacement of the free end. The constant of proportionality k N m^{-1} is known as the **stiffness** of the spring. Calculate the work done in stretching a spring with stiffness k from displacement x_1 m to displacement x_2 m $(x_2 > x_1)$ given that the work done (W) is the product of force and displacement.

Your solution

Answer

The restoring force varies during the displacement. So the work done during the extension cannot be determined from a single simple product.

Consider a small element Δx of the extension beyond an arbitrary displacement x. The element is sufficiently small that the force during the displacement can be regarded as constant and equal to the force at displacement x is kx. So the work done ΔW in extending the spring from displacement x to displacement $x + \Delta x$ is approximately $kx\Delta x$.

Using the idea of integration as a limit of a sum, in this case as Δx tends to zero,

$$W = \int_{x_1}^{x_2} kx\,dx = \left[\frac{1}{2}kx^2\right]_{x_1}^{x_2} = \frac{1}{2}k(x_2^2 - x_1^2)$$

Exercises

1. Evaluate (a) $\displaystyle\int_0^1 x^2\,dx$, (b) $\displaystyle\int_2^3 \frac{1}{x^2}\,dx$ (c) $\displaystyle\int_1^2 \mathrm{e}^x\,dx$ (d) $\displaystyle\int_{-1}^1 (1+t^2)\,dt$

2. Find (a) $\displaystyle\int_0^{\pi/3} \cos 2x\,dx$ (b) $\displaystyle\int_0^{\pi} \sin x\,dx$ (c) $\displaystyle\int_1^3 \mathrm{e}^{2t}\,dt$

Answers

1. (a) $\dfrac{1}{3}$ (b) $\dfrac{1}{6}$ (c) $\mathrm{e}^2 - \mathrm{e}^1 = 4.671$ (d) 2.667

2 (a) $\sqrt{3}/4 = 0.4330$ (b) 2 (c) 198.019

Engineering Example 2

Torsion of a mild-steel bar

Introduction

For materials such as mild-steel, the relationship between applied shear stress and shear strain (deformation) can be described as follows.

- For small values of the shear strain, the shear stress (τ) and shear strain (ω) are proportional to one another, i.e.

$$\omega = \frac{1}{G} \times \tau \tag{1}$$

  (where G is the *shear modulus*). This is known as **elastic behaviour**.

- There is a maximum shear stress that the material is capable of supporting. If the shear strain is increased further, the shear stress remains roughly constant. This is known as **plastic behaviour**.

Figure 3 summarises the relationship between shear stress and shear strain; the point (ω_Y, τ_Y) is known as the **yield point**.

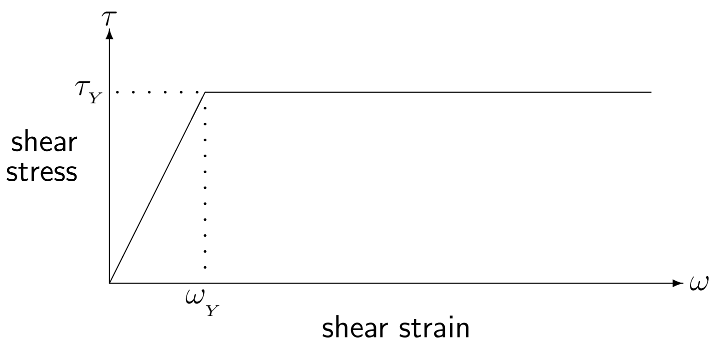

Figure 3

Now suppose that one end of a bar of circular cross section is twisted through an angle θ, then the shear strain on the surface is given by

$$\omega_S = \frac{R\,\theta}{L} \tag{2}$$

(where R and L are the radius and length of the bar respectively), while the shear strain, at a distance r from the central core, is given by

$$\omega = \frac{r\,\theta}{L} \tag{3}$$

The torque transmitted by a bar is given by the integral

$$T = \int_0^R 2\pi\, r^2\, \tau(r)\, dr \tag{4}$$

As the shear strain is a function of distance from the central axis of the bar, it may be that the shear strain on the surface is greater than the critical shear strain $\omega_{_Y}$. In this scenario the shear stress is given by

$$\tau = \begin{cases} \dfrac{\tau_Y}{\omega_{_Y}}\,\omega & \omega \leq \omega_{_Y} \\ \tau_Y & \omega > \omega_{_Y} \end{cases} \tag{5}$$

i.e. the regions near the central axis exhibit elasticity, but in those regions near the surface the elastic limit has been exceeded and the metal exhibits plasticity (see Figure 4).

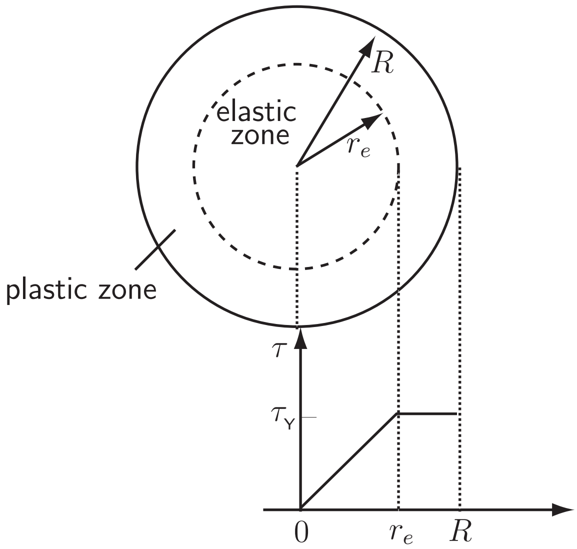

Figure 4

Problem in words

Find an expression for the torque transmitted by a bar as a function of the angle θ through which one end is turned.

Mathematical statement of problem

Using Equations (3) to (5), find a formula for T in terms of the variable θ.

Mathematical analysis

Substituting (3) into (5)

$$\begin{aligned} \tau &= \begin{cases} \dfrac{\tau_Y}{\omega_{_Y}}\dfrac{r\,\theta}{L} & \dfrac{r\,\theta}{L} \leq \omega_{_Y} \\ \tau_Y & \dfrac{r\,\theta}{L} > \omega_{_Y} \end{cases} \\ &= \begin{cases} \dfrac{\tau_Y}{\omega_{_Y}}\dfrac{r\,\theta}{L} & r \leq \dfrac{L\,\omega_{_Y}}{\theta} = r_e \\ \tau_Y & r > \dfrac{L\,\omega_{_Y}}{\theta} = r_e \end{cases} \end{aligned}$$

For small values of θ, $r_e \geq R$ so that the whole of the bar will be in the elastic region, i.e.

$$\tau = \frac{\tau_Y}{\omega_Y}\frac{r\,\theta}{L}$$

Now (4) becomes

$$T = \int_0^R 2\pi\, r^2\,\frac{\tau_Y}{\omega_Y}\frac{r\,\theta}{L}\,dr = 2\pi\,\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\int_0^R r^3\,dr = 2\pi\,\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\left[\frac{r^4}{4}\right]_0^R = \frac{\pi}{2}\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\,R^4 \qquad (6)$$

i.e. the torque is directly proportional to the twist, θ.

For larger θ, $r_e < R$, so that (4) becomes

$$\begin{aligned} T &= \int_0^{r_e} 2\pi\, r^2\,\frac{\tau_Y}{\omega_Y}\frac{r\,\theta}{L}\,dr + \int_{r_e}^R 2\pi\, r^2\,\tau_Y\,dr \\ &= 2\pi\,\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\int_0^{r_e} r^3\,dr + 2\pi\,\tau_Y\int_{r_e}^R r^2\,dr \\ &= 2\pi\,\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\left[\frac{r^4}{4}\right]_0^{r_e} + 2\pi\,\tau_Y\left[\frac{r^3}{3}\right]_{r_e}^R \\ &= \frac{\pi}{2}\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\,r_e^4 + \frac{2\pi}{3}\,\tau_Y\left(R^3 - r_e^3\right) \end{aligned}$$

But $r_e = L\,\omega_Y/\theta$, so

$$\begin{aligned} T &= \frac{\pi}{2}\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\frac{L^4\,\omega_Y^4}{\theta^4} + \frac{2\pi}{3}\,\tau_Y\,R^3 - \frac{2\pi}{3}\,\tau_Y\,\frac{L^3\,\omega_Y^3}{\theta^3} \\ &= \frac{2\pi}{3}\,\tau_Y\,R^3 + \pi\left(\frac{1}{2}\,\tau_Y - \frac{2}{3}\,\tau_Y\right)\frac{L^3\,\omega_Y^3}{\theta^3} \\ &= \frac{2\pi}{3}\,\tau_Y\,R^3 - \frac{\pi}{6}\,\tau_Y\,\frac{L^3\,\omega_Y^3}{\theta^3} \qquad (7) \end{aligned}$$

Equation (6) will apply when $r_e \geq R$, i.e. $(L\,\omega_Y/\theta) \geq R$ or $\theta \leq (L\,\omega_Y/R)$, so that combining (6) and (7) gives overall

$$T = \begin{cases} \dfrac{\pi}{2}\dfrac{\tau_Y}{\omega_Y}\dfrac{\theta}{L}\,R^4 & \theta \leq \dfrac{L\,\omega_Y}{R} \\[2ex] \dfrac{2\pi}{3}\,\tau_Y\,R^3 - \dfrac{\pi}{6}\,\tau_Y\,\dfrac{L^3\,\omega_Y^3}{\theta^3} & \theta > \dfrac{L\,\omega_Y}{R} \end{cases} \qquad (8)$$

Interpretation and further comment

At the critical value of θ, i.e. when the outer edge begins to exhibit plasticity, both formulae in (8) give

$$T_{crit} = \frac{\pi}{2}\,\tau_Y\,R^3$$

Furthermore, the first derivatives are both

$$\frac{dT}{d\theta} = \frac{\pi}{2}\frac{\tau_Y}{\omega_Y}\frac{R^4}{L}$$

i.e. the curves join smoothly.

The second derivatives, though, are not equal (zero in one case). In the theoretical limit as $\theta \to \infty$

$$T = \frac{2\pi}{3}\tau_{_Y} R^3$$

so this is the total torsional torque which can be carried by the bar. (The critical torque above is three-quarters of this value.) However, clearly $\theta \to \infty$ is merely a theoretical limit since the bar would, in fact, shear at a finite value of θ.

3. Some integrals with infinite limits

On occasions, and notably when dealing with Laplace and Fourier transforms, you will come across integrals in which one of the limits is infinite. We avoid a rigorous treatment of such cases here and instead give some commonly occurring examples.

Example 11

Find the definite integral of e^{-x} from 0 to ∞; that is, find $\displaystyle\int_0^\infty e^{-x}\, dx$.

Solution

The integral is found in the normal way: $\displaystyle\int_0^\infty e^{-x}\, dx = \Big[-e^{-x} \Big]_0^\infty$

There is no difficulty in evaluating the square bracket at the lower limit. We obtain simply $-e^{-0} = -1$. At the upper limit we must examine the behaviour of $-e^{-x}$ as x gets infinitely large. This is where it is important that you are familiar with the properties of the exponential function. If you refer to the graph (Figure 5) you will see that as x tends to infinity e^{-x} tends to zero.

Consequently the contribution to the integral from the upper limit is zero. So

$$\begin{aligned}\int_0^\infty e^{-x}\mathrm{d}x &= \Big[-e^{-x}\Big]_0^\infty \\ &= (-e^{-\infty}) - (-e^{-0}) \\ &= (0) \quad - \quad (-e^{-0}) \\ &= 1\end{aligned}$$

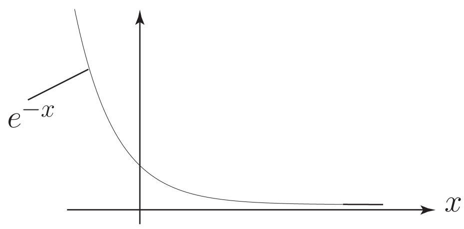

Figure 5

Thus the value of $\displaystyle\int_0^\infty e^{-x}\, dx$ is 1.

Another way of achieving this result is as follows:

We change the infinite limit to a finite limit, b, say and then examine the behaviour of the integral as b tends to infinity, written as

$$\int_0^{\infty} \mathrm{e}^{-x}\,dx = \lim_{b\to\infty}\int_0^{b} \mathrm{e}^{-x}\,dx$$

Now, $\displaystyle\int_0^{b} \mathrm{e}^{-x}\,dx = \Big[-\mathrm{e}^{-x}\Big]_0^b = \left(-\mathrm{e}^{-b}\right) - \left(-\mathrm{e}^{-0}\right) = -\mathrm{e}^{-b} + 1$

Then as b tends to infinity $-\mathrm{e}^{-b}$ tends to zero, and the resulting integral has the value 1, as before. Many integrals having infinite limits cannot be evaluated in a simple way like this, and many cannot be evaluated at all. Fortunately, most of the integrals you will meet will exhibit the sort of behaviour seen in the last example.

Exercise

Evaluate (a) $\displaystyle\int_1^{\infty} \mathrm{e}^{-x}\,dx$ (b) $\displaystyle\int_0^{\infty} \mathrm{e}^{-2x}\,dx$ (c) $\displaystyle\int_2^{\infty} \mathrm{e}^{-3x}\,dx$ (d) $\displaystyle\int_1^{\infty} \frac{4}{t^2}\,dt$

Answer

(a) $\mathrm{e}^{-1} \sim 0.368$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}\mathrm{e}^{-6} = 0.0008$ (4 d.p.) (d) 4