# Case Study: Complex Impedance

<div style="float:right">**12.6**</div>

## Electronic Filters

Electronic filters are used widely, for example in audio equipment to correct for imperfections in microphones or loudspeakers, or to introduce special effects. The purpose of a filter is to produce an alternating current (a.c.) output voltage that varies with the frequency of the input voltage. A filter must have at least one component which has an impedance that varies with frequency. The **impedance** is given by the time dependent ratio of 'voltage across the component' to 'current through the component'. This means that a filter must contain at least one inductance or capacitance. An **inductor** consists of a large number of coils of wire. When the current $i$ flowing through an inductor changes, the associated magnetic field changes and produces a voltage v across the inductor which is proportional to the rate of change of the current. The constant of proportionality (**inductance**) is given the symbol $L$.

In electronics, it is usual to use lower case symbols for the time varying quantities. The standard representations for a.c. electronic signals are

$$v = V_0 e^{j\omega t} \text{ and } i = I_0 e^{j\omega t}$$

where $V_0$ is the (real) amplitude of the a.c. voltage and $I_0$ is the (real) amplitude of the a.c. current and $j = \sqrt{-1}$.
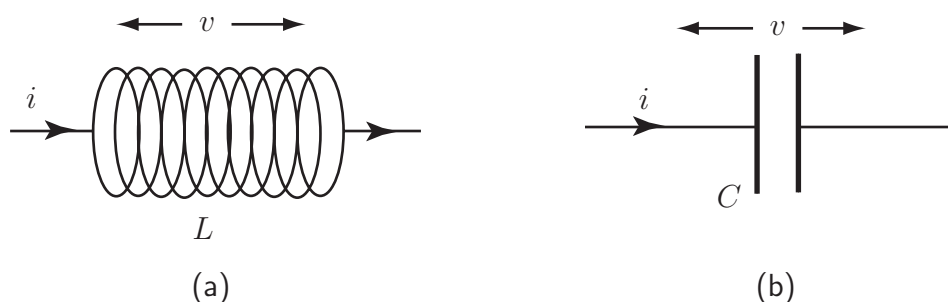


**Figure 33**: (a) an inductor (b) a capacitor

An inductor (see Figure 33) gives rise to an a.c. voltage

$$v = L\frac{di}{dt} = j\omega L i$$

Hence $v/i = jwL$ is the impedance of the inductor. The purely imaginary quantity, $jwL$, is called the **reactance** of the inductor. Usually a coil of wire forming an inductor also has resistance but this can be designed or assumed to be negligible. A **capacitor** consists of two conducting plates separated by a thin insulator. The charge ($q$) on the plates is proportional to the voltage ($v$) between the plates. The constant of proportionality (**capacitance**) is given the symbol $C$. So $q = Cv$. The current ($i$) into the capacitor is equal to the rate of change of the charge on the capacitor i.e.

$$i = \frac{dq}{dt} = C\frac{dv}{dt} = j\omega Cv.$$

Hence, for a capacitor, the impedance $Z_c = v/i = 1/jwC$. This purely imaginary quantity is also a reactance. Because of Ohm's law ($v = iR$), a resistance $R$ provides a constant (real) contribution of $R$ to the impedance of a circuit. If two resistors $R_1$ and $R_2$ are in series the same current passes through both of them and the combined resistance is $R_1 + R_2$. In the circuit shown in Figure 34 (consider the left-hand representation of this circuit first but note that the right-hand version is equivalent), the input voltage across both resistors and the output voltage across $R_2$ are related by
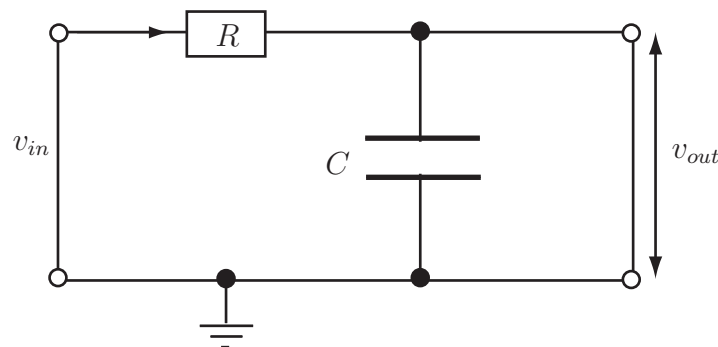
$$v_{in} = i(R_1 + R_2) \quad \text{and} \quad v_{out} = iR_2 \quad \text{so} \quad \frac{v_{out}}{v_{in}} = \frac{R_2}{R_1 + R_2}.$$

Such a circuit is called a **potential divider**.
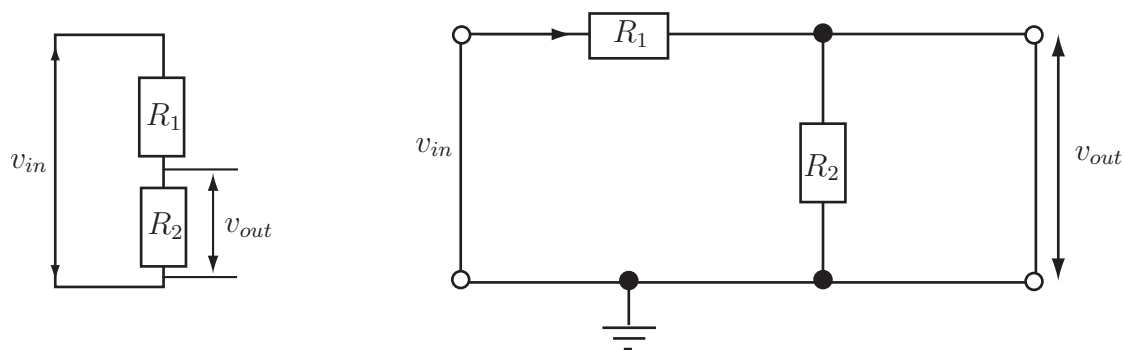


**Figure 34**: Two representations of a potential divider circuit

Now consider this circuit with the resistor $R_2$ replaced by a capacitor $C$ as in Figure 35.



**Figure 35**: Low pass filter circuit containing a resistor and a capacitor

If $R_1$ is replaced by $R$ and $R_2$ by $Z_C = 1/jwC$, in the relevant expression for the potential divider circuit, then

$$\frac{v_{out}}{v_{in}} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

The square of the magnitude of the voltage ratio is given by multiplying the existing complex expression by its complex conjugate, i.e.

$$\left|\frac{v_{out}}{v_{in}}\right|^2 = \frac{1}{(1 + j\omega RC)(1 - j\omega RC)} = \frac{1}{(1 + \omega^2 R^2 C^2)}$$

Figure 36 shows a plot of the magnitude of the voltage ratio as a function of $\omega$, i.e. the **frequency response** for $R = 10$ $\Omega$ and $C = 1$ $\mu$F (i.e. $10^{-6}$F). Note that the magnitude of the output voltage is close to that of the input voltage at low frequencies but decreases rapidly as frequency increases. This is an ideal low pass filter response.
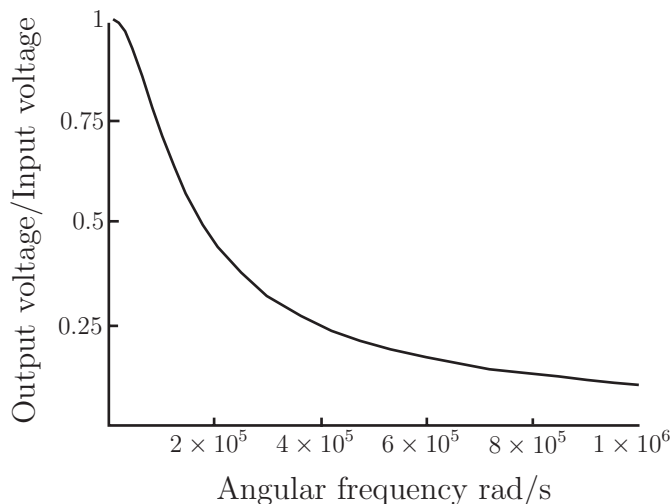


**Figure 36**: Frequency response of a low pass filter

**Engineering problem stated in words**
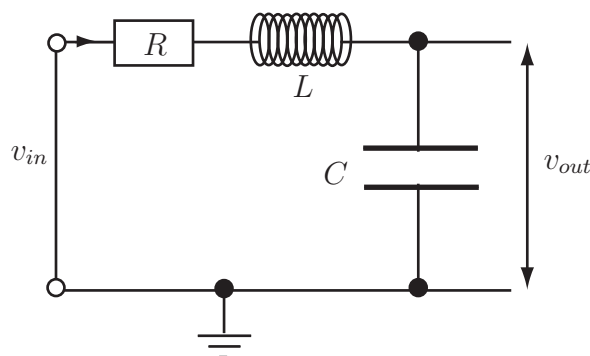


**Figure 37**: An $LC$ filter circuit

Plot the frequency response of the $LC$ filter circuit shown in Figure 37 if $R = 10$ $\Omega$, $L = 0.1$ mH (i.e. $10^{-4}$H) and $C = 1$ $\mu$F. After plotting the response for two values of $R$ below 10 $\Omega$, comment on the way in which the response varies as $R$ varies. Identify the frequency for which the response is maximum.

**Engineering problem expressed mathematically**

(a) Noting that the resistor and inductor are in series, replace $R_1$ by $(R + jwL)$ and $R2$ by $1/jwC$ in the equation $\dfrac{v_{out}}{v_{in}} = \dfrac{R_2}{R_1 + R_2}$

(b) Derive an expression for $\left| \dfrac{v_{out}}{v_{in}} \right|^2$

(c) Hence plot $\left| \dfrac{v_{out}}{v_{in}} \right|$ as a function of $\omega$ for $R = 10$ $\Omega$.

(d) Plot $\left|\dfrac{v_{out}}{v_{in}}\right|$ for two further values of $R < 10\ \Omega$ (e.g. $5\ \Omega$ and $2\ \Omega$).

(e) Find an expression for the value of $\omega = \omega_{res}$ at which $\left|\dfrac{v_{out}}{v_{in}}\right|$ is maximum.
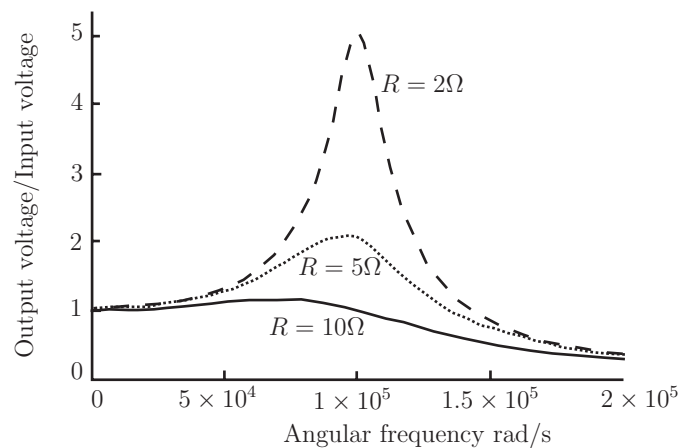
## Mathematical analysis

(a) The substitutions $R_1 \to (R + jwL)$ and $R_2 \to 1/jwC$ in the equation

$$\frac{v_{out}}{v_{in}} = \frac{R_2}{R_1 + R_2} \quad \text{yield} \quad \frac{v_{out}}{v_{in}} = \frac{1/j\omega C}{R + j\omega L + 1/j\omega C} = \frac{1}{(1 - \omega^2 LC + j\omega RC)}$$

(b) Multiplying by the complex conjugate of the denominator

$$\left|\frac{v_{out}}{v_{in}}\right|^2 = \frac{1}{(1 - \omega^2 LC + j\omega RC)(1 - \omega^2 LC - j\omega RC)} = \frac{1}{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2}$$

(c) See the solid line in Figure 38.



**Figure 38**: Frequency response of $LC$ filter

(d) See the other broken lines in Figure 38.

There is a peak in the voltage output, which can exceed the voltage input by a considerable amount. It is particularly noticeable for small values of the resistance and decreases as the resistance increases.

(e) $\left|\dfrac{v_{out}}{v_{in}}\right|$ will be maximum when the first term in the denominator is zero (the other term is always positive for $\omega > 0$) i.e. when

$$\omega = \omega_{res} = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f_{res} = \frac{\omega_{res}}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

The corresponding frequency is known as the resonant frequency of the circuit.

## Additional comment

The resonant behaviour depicted in Figure 38 is found in certain vibrating systems as well as electronic circuits. This gives rise to an electrical analogy for such mechanical systems and will be explored further after HELM 19 on differential equations.