Curvature

12.4

Introduction

Curvature is a measure of how sharply a curve is turning. At a particular point along the curve a tangent line can be drawn; this tangent line making an angle ψ with the positive x-axis. Curvature is then defined as the magnitude of the rate of change of ψ with respect to the measure of length on the curve - the arc length s. That is

$$\text{Curvature } = \left|\frac{d\psi}{ds}\right|$$

In this Section we examine the concept of curvature and, from its definition, obtain more useful expressions for curvature when the equation of the curve is expressed either in Cartesian form $y = f(x)$ or in parametric form $x = x(t) \quad y = y(t)$. We show that a circle has a constant value for the curvature, which is to be expected, as the tangent line to a circle turns equally quickly irrespective of the position on the circle. For all curves, except circles, other than a circle, the curvature will depend upon position, changing its value as the curve twists and turns.

Prerequisites

Before starting this Section you should ...

- understand the geometrical interpretation of the derivative
- be able to differentiate standard functions
- be able to use the parametric description of a curve

Learning Outcomes

On completion you should be able to ...

- understand the concept of curvature
- calculate curvature when the curve is defined in Cartesian form or in parametric form

1. Curvature

Curvature is a measure of how quickly a tangent line turns as the contact point moves along a curve. For example, consider a simple parabola, with equation $y = x^2$. Its graph is shown in Figure 27.

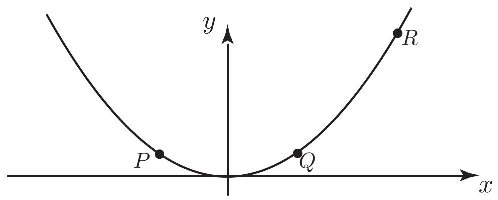


Figure 27

It is obvious, geometrically, that the tangent lines to this curve turn 'more quickly' between P and Q than between Q and R. It is the purpose of this Section to give, a quantitative measure of this rate of 'turning'.

If we change from a parabola to a circle, (centred on the origin, of radius 1), we can again consider how quickly the tangent lines turn as we move along the curve. See Figure 28. It is immediately clear that the tangent lines to a circle turn equally quickly no matter where located on the circle.

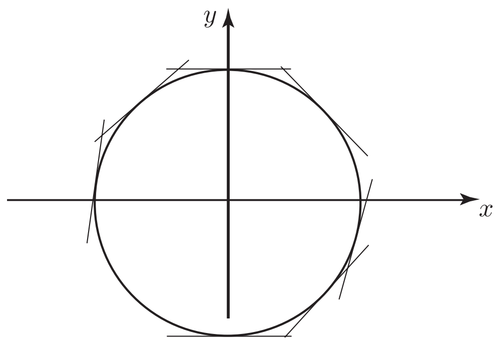


Figure 28

However, if we consider two circles with the same centre but different radii, as in Figure 29, it is again obvious that the smaller circle 'bends' more tightly than the larger circle and we say it has a larger curvature. Athletes who run the 200 metres find it easier to run in the outside lanes (where the curve turns less sharply) than in the inside lanes.

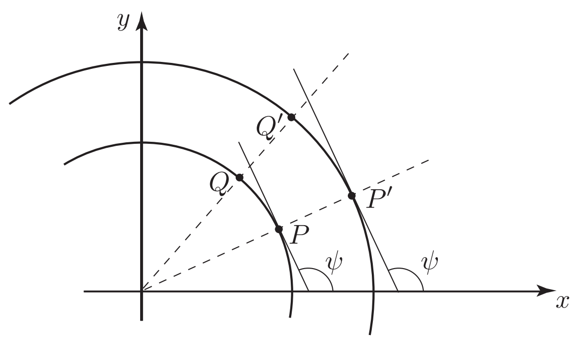


Figure 29

On the two circle diagram (Figure 29) we have drawn tangent lines at P and P'; both lines make an angle ψ (greek letter psi) with the positive x-axis. We need to measure how quickly the angle

ψ changes as we move along the curve. As we move from P to Q (inner circle), or from P' to Q' (outer circle), the angle ψ changes by the same amount. However, the distance traversed on the inner circle is less than the distance traversed on the outer circle. This suggests that a measure of curvature is:

curvature is the magnitude of the rate of change of ψ with respect to the distance moved along the curve.

We shall denote the curvature by the Greek letter κ (kappa).
So

$$\kappa = \left|\frac{d\psi}{ds}\right|$$

where s is the measure of arc-length along a curve. This rather odd-looking derivative needs converting to involve the variable x if the equation of the curve is given in the usual form $y = f(x)$. As a preliminary we note that

$$\frac{d\psi}{ds} = \frac{d\psi}{dx}\bigg/\frac{ds}{dx}$$

We now obtain expressions for the derivatives $\dfrac{d\psi}{dx}$ and $\dfrac{ds}{dx}$ in terms of the derivatives of $f(x)$.
Consider Figure 30 below.

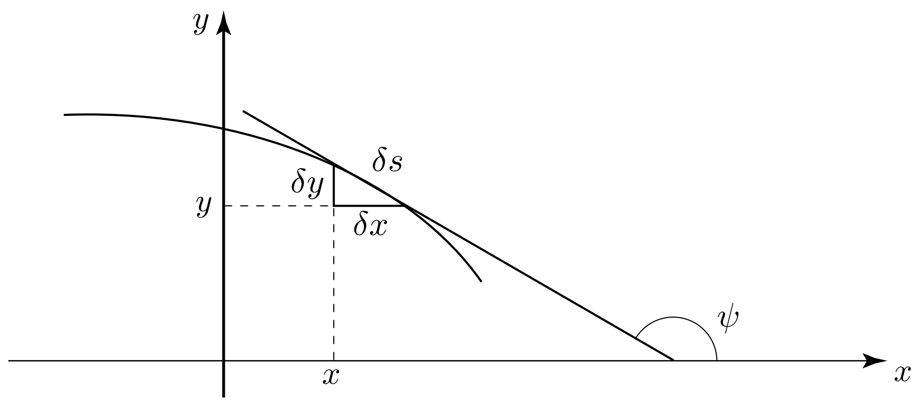


Figure 30

Small increments in the x- and y-directions have been denoted by δx and δy respectively. The hypotenuse on this 'small triangle' is δs which is the change in arc-length along the curve.
From Pythagoras' theorem:

$$\delta s^2 = \delta x^2 + \delta y^2$$

so

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2 \qquad \text{so that} \qquad \frac{\delta s}{\delta x} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

In the limit as the increments get smaller and smaller, we write this relation in derivative form:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

However, as $y = f(x)$ is the equation of the curve we obtain

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{df}{dx}\right)^2} = (1 + [f'(x)]^2)^{1/2}$$

We also know the relation between the angle ψ and the derivative $\dfrac{df}{dx}$:

$$\frac{df}{dx} = \tan\psi$$

so differentiating again:

$$\frac{d^2 f}{dx^2} = \sec^2\psi \frac{d\psi}{dx} = (1 + \tan^2\psi)\frac{d\psi}{dx}$$

$$= (1 + [f'(x)]^2)\frac{d\psi}{dx}$$

Inverting this relation:

$$\frac{d\psi}{dx} = \frac{f''(x)}{(1 + [f'(x)]^2)}$$

and so, finally, the curvature is given by

$$\kappa = \left|\frac{d\psi}{ds}\right| = \left|\frac{d\psi}{dx}\Big/\frac{ds}{dx}\right| = \left|\frac{f''(x)}{(1 + [f'(x)]^2)^{3/2}}\right|$$

Key Point 6

Curvature

At each point on a curve, with equation $y = f(x)$, the tangent line turns at a certain rate.

A measure of this rate of turning is the **curvature** κ defined by

$$\kappa = \left|\frac{f''(x)}{(1 + [f'(x)]^2)^{3/2}}\right|$$

Obtain the curvature of the parabola $y = x^2$.

First calculate the derivatives of $f(x)$:

Your solution

$f(x) =$ $\qquad \dfrac{df}{dx} =$ $\qquad \dfrac{d^2f}{dx^2} =$

Answer

$f(x) = x^2 \qquad \dfrac{df}{dx} = 2x \qquad \dfrac{d^2f}{dx^2} = 2$

Now find an expression for the curvature:

Your solution

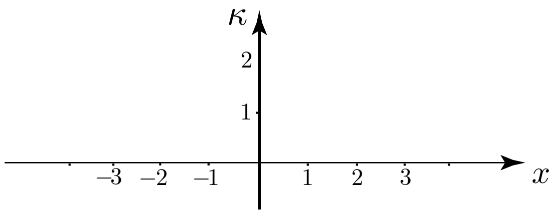
$\kappa =$

Answer

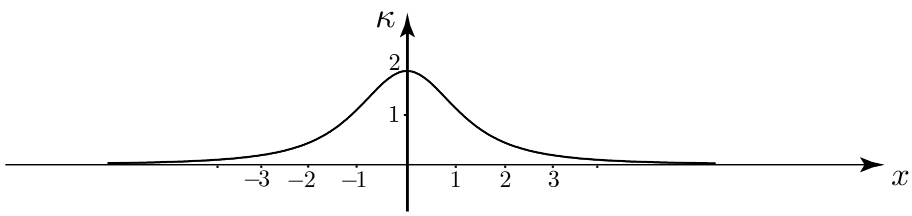
$$\kappa = \left| \frac{f''(x)}{[1 + [f'(x))]^2]^{3/2}} \right| = \frac{2}{[1 + 4x^2]^{3/2}}$$

Finally, plot the curvature κ as a function of x:

Your solution



Answer



The figure above supports what we have already argued:

- Close to $x = 0$ the parabola turns sharply (near $x = 0$ the curvature κ is relatively, large).
- Further away from $x = 0$ the curve is more 'gentle' (in these regions κ is small).

In general, the curvature κ is a function of position. However, from what we have said earlier, we expect the curvature to be a constant for a given circle but to increase as the radius of the circle decreases. This can now be checked directly.

Example 5

Find the curvature of $y = (a^2 - x^2)^{1/2}$ (this is the equation of the upper half of a circle centred at the origin of radius a).

Solution

Here $f(x) = (a^2 - x^2)^{\frac{1}{2}}$

$$\frac{df}{dx} = \frac{-x}{(a^2 - x^2)^{\frac{1}{2}}} \qquad \frac{d^2 f}{dx^2} = \frac{-a^2}{(a^2 - x^2)^{\frac{3}{2}}}$$

$$\therefore \qquad 1 + [f'(x)]^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{r^2}{a^2 - x^2}$$

$$\therefore \qquad \kappa = \left| \frac{\dfrac{-a^2}{(a^2 - x^2)^{3/2}}}{\left[\dfrac{a^2}{a^2 - x^2}\right]^{3/2}} \right| = \frac{1}{a}$$

For a circle of radius a, the curvature is constant, with value $\dfrac{1}{a}$.

The value of κ (at any particular point on the curve, i.e. at a particular value of x) indicates how sharply the curve is turning. What this result states is that, for a circle, the curvature is inversely related to the radius. The bigger the radius, the smaller the curvature; precisely what we predicted.

2. Curvature for parametrically defined curves

An expression for the curvature is also available if the curve is described parametrically:

$$x = g(t) \qquad y = h(t) \qquad t_0 \leq t \leq t_1$$

We remember the basic formulae connecting derivatives

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \frac{d^2 y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

where, as usual $\dot{x} \equiv \dfrac{dx}{dt}$, $\quad \ddot{x} \equiv \dfrac{d^2 x}{dt^2}$ etc.

Then

$$\kappa = \left| \frac{f''(x)}{\{1 + [f'(x)]^2\}^{3/2}} \right| = \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3 \left[1 + \left(\dfrac{\dot{y}}{\dot{x}}\right)^2\right]^{3/2}} \right|$$

$$= \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \right|$$

Key Point 7

The formula for curvature in parametric form is $\kappa = \left| \dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \right|$

Task

An ellipse is described parametrically by the equations

$$x = 2\cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi$$

Obtain an expression for the curvature κ and find where the curvature is a maximum or a minimum.

First find $\dot{x},\ \dot{y},\ \ddot{x},\ \ddot{y}$:

Your solution

$\dot{x} =$ $\qquad$ $\dot{y} =$ $\qquad$ $\ddot{x} =$ $\qquad$ $\ddot{y} =$

Answer

$\dot{x} = -2\sin t \qquad \dot{y} = \cos t \qquad \ddot{x} = -2\cos t \qquad \ddot{y} = -\sin t$

Now find κ:

Your solution

$\kappa =$

Answer

$$\kappa = \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \right| = \left| \frac{2\sin^2 t + 2\cos^2 t}{[4\sin^2 t + \cos^2 t]^{3/2}} \right| = \frac{2}{[1 + 3\sin^2 t]^{3/2}}$$

Find maximum and minimum values of κ **by inspection** of the expression for κ:

Your solution

max $\kappa =$ $\qquad$ min $\kappa =$

Answer

Denominator is max when $t = \pi/2$. This gives minimum value of $\kappa = 1/4$,
Denominator is min when $t = 0$. This gives maximum value of $\kappa = 2$.