

The Newton-Raphson Method

12.3



Introduction

This Section is concerned with the problem of “root location”; i.e. finding those values of x which satisfy an equation of the form $f(x) = 0$. An initial estimate of the root is found (for example by drawing a graph of the function). This estimate is then improved using a technique known as the Newton-Raphson method, which is based upon a knowledge of the tangent to the curve near the root. It is an “iterative” method in that it can be used repeatedly to continually improve the accuracy of the root.



Prerequisites

Before starting this Section you should ...

- be able to differentiate simple functions
- be able to sketch graphs



Learning Outcomes

On completion you should be able to ...

- distinguish between simple and multiple roots
- estimate the root of an equation by drawing a graph
- employ the Newton-Raphson method to improve the accuracy of a root

1. The Newton-Raphson method

We first remind the reader of some basic notation: If $f(x)$ is a given function the value of x for which $f(x) = 0$ is called a **root** of the equation or **zero** of the function. We also distinguish between various types of roots: simple roots and multiple roots. Figures 21 - 23 illustrate some common examples.

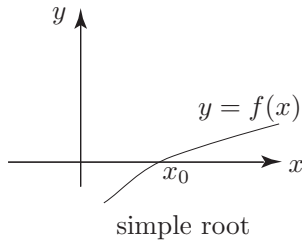


Figure 21

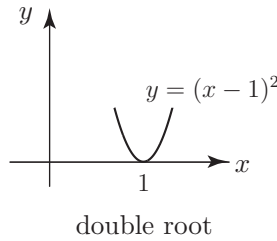


Figure 22

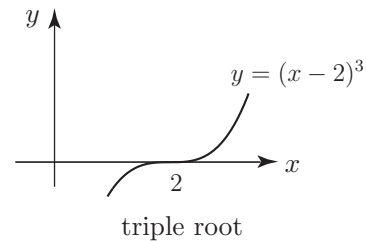


Figure 23

More precisely; a root x_0 is said to be:

a **simple root** if $f(x_0) = 0$ and $\left. \frac{df}{dx} \right|_{x_0} \neq 0$.

a **double root** if $f(x_0) = 0$, $\left. \frac{df}{dx} \right|_{x_0} = 0$ and $\left. \frac{d^2f}{dx^2} \right|_{x_0} \neq 0$, and so on.

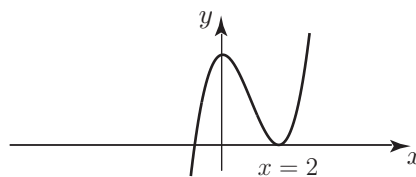
In this Section we shall concentrate on the location of simple roots of a given function $f(x)$.



Given graphs of the functions (a) $f(x) = x^3 - 3x^2 + 4$, (b) $f(x) = 1 + \sin x$ classify the roots into simple or multiple.

Your solution

(a) $f(x) = x^3 - 3x^2 + 4$: The negative root is: and the positive root is:

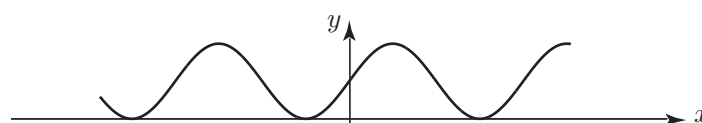


Answer

The negative root is simple and the positive root is double.

Your solution

(b) $f(x) = 1 + \sin x$: Each root is a root



Answer

Each root is a double root.

2. Finding roots of the equation $f(x) = 0$

A first investigation into the roots of $f(x)$ might be graphical. Such an analysis will supply information as to the approximate location of the roots.

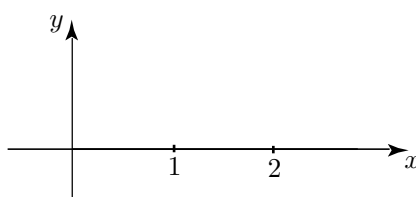


Sketch the function

$$f(x) = x - 2 + \ln x \quad x > 0$$

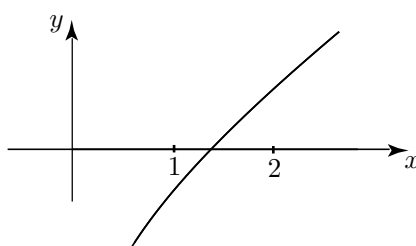
and estimate the value of the root.

Your solution



An estimate of the root is:

Answer



A simple root is located near 1.5

One method of obtaining a better approximation is to halve the interval $1 \leq x \leq 2$ into $1 \leq x \leq 1.5$ and $1.5 \leq x \leq 2$ and test the **sign** of the function at the end-points of these new regions. We find

x	$f(x)$
1	< 0
1.5	< 0
2	> 0

so a root must lie between $x = 1.5$ and $x = 2$ because the sign of $f(x)$ changes between these values and $f(x)$ is a continuous curve. We can repeat this procedure and divide the interval $(1.5, 2)$ into the two new intervals $(1.5, 1.75)$ and $(1.75, 2)$ and test again. This time we find

x	$f(x)$
1.5	< 0
1.75	> 0
2.0	> 0

so a root lies in the interval $(1.5, 1.75)$. It is obvious that proceeding in this way will give a smaller and smaller interval in which the root must lie. But can we do better than this rather laborious bisection procedure? In fact there are many ways to improve this numerical search for the root. In this Section we examine one of the best methods: the **Newton-Raphson** method.

To derive the method we examine the general characteristics of a curve in the neighbourhood of a simple root. Consider Figure 24 showing a function $f(x)$ with a simple root at $x = x^*$ whose value is required. Initial analysis has indicated that the root is approximately located at $x = x_0$. The aim is to provide a better estimate to the location of the root.

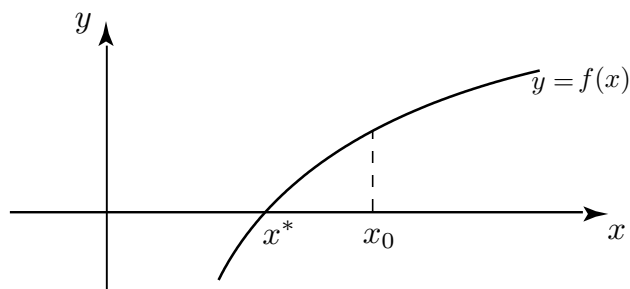


Figure 24

The basic premise of the Newton-Raphson method is the assumption that the curve in the close neighbourhood of the simple root at x^* is approximately a straight line. Hence if we draw the tangent to the curve at x_0 , this tangent will intersect the x -axis at a point closer to x^* than is x_0 : see Figure 25.

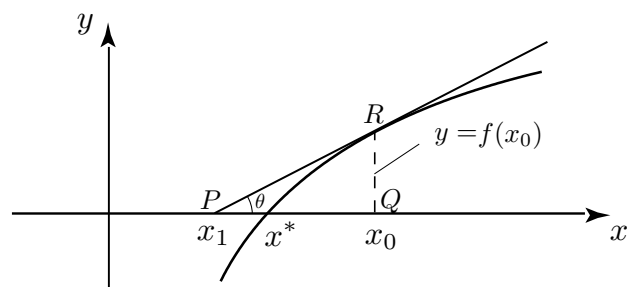


Figure 25

From the geometry of this diagram we see that

$$x_1 = x_0 - PQ$$

But from the right-angled triangle PQR we have

$$\frac{RQ}{PQ} = \tan \theta = f'(x_0)$$

and so
$$PQ = \frac{RQ}{f'(x_0)} = \frac{f(x_0)}{f'(x_0)} \quad \therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If $f(x)$ has a simple root near x_0 then a closer estimate to the root is x_1 where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This formula can be used iteratively to get closer and closer to the root, as summarised in Key Point 5:



Key Point 5

Newton-Raphson Method

If $f(x)$ has a simple root near x_n then a closer estimate to the root is x_{n+1} where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the **Newton-Raphson iterative formula**. The iteration is begun with an initial estimate of the root, x_0 , and continued to find x_1, x_2, \dots until a suitably accurate estimate of the position of the root is obtained. This is judged by the convergence of x_1, x_2, \dots to a fixed value.



Example 4

$f(x) = x - 2 + \ln x$ has a root near $x = 1.5$. Use the Newton-Raphson method to obtain a better estimate.

Solution

Here $x_0 = 1.5$, $f(1.5) = -0.5 + \ln(1.5) = -0.0945$

$$f'(x) = 1 + \frac{1}{x} \quad \therefore \quad f'(1.5) = 1 + \frac{1}{1.5} = \frac{5}{3}$$

Hence using the formula:

$$x_1 = 1.5 - \frac{(-0.0945)}{(1.6667)} = 1.5567$$

The Newton-Raphson formula can be used again: this time beginning with 1.5567 as our estimate:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5567 - \frac{f(1.5567)}{f'(1.5567)} = 1.5567 - \frac{\{1.5567 - 2 + \ln(1.5567)\}}{\left\{1 + \frac{1}{1.5567}\right\}} \\ &= 1.5567 - \frac{\{-0.0007\}}{\{1.6424\}} = 1.5571 \end{aligned}$$

This is in fact the correct value of the root to 4 d.p., which calculating x_3 would confirm.



The function $f(x) = x - \tan x$ has a simple root near $x = 4.5$. Use one iteration of the Newton-Raphson method to find a more accurate value for the root.

First find $\frac{df}{dx}$:

Your solution

$$\frac{df}{dx} =$$

Answer

$$\frac{df}{dx} = 1 - \sec^2 x = -\tan^2 x$$

Now use the formula $x_1 = x_0 - f(x_0)/f'(x_0)$ with $x_0 = 4.5$ to obtain x_1 :

Your solution

$$\begin{aligned} f(4.5) &= 4.5 - \tan(4.5) = \\ f'(4.5) &= 1 - \sec^2(4.5) = -\tan^2(4.5) = \\ x_1 &= 4.5 - \frac{f(4.5)}{f'(4.5)} = \end{aligned}$$

Answer

$$f(4.5) = -0.1373, \quad f'(4.5) = -21.5048$$

$$\therefore x_1 = 4.5 - \frac{0.1373}{21.5048} = 4.4936.$$

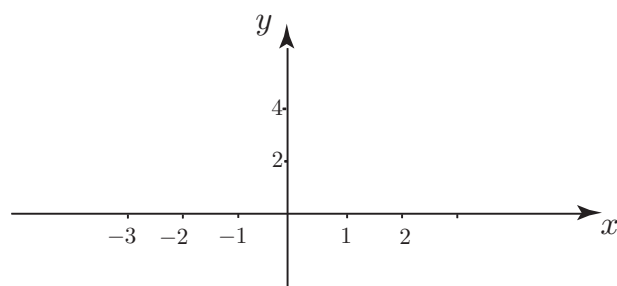
As the value of x_1 has changed little from $x_0 = 4.5$ we can expect the root to be 4.49 to 3 d.p.

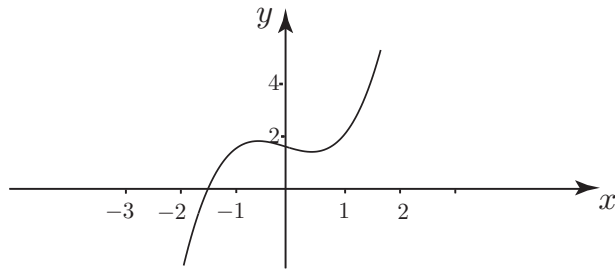


Sketch the function $f(x) = x^3 - x + 3$ and confirm that there is a simple root between $x = -2$ and $x = -1$. Use $x_0 = -2$ as an initial estimate to obtain the value to 2 d.p.

First sketch $f(x) = x^3 - x + 3$ and identify a root:

Your solution



Answer

Clearly a simple root lies between $x = -2$ and $x = -1$.

Now use one iteration of Newton-Raphson to improve the estimate of the root using $x_0 = -2$:

Your solution

$$f(x) = \quad \quad \quad f'(x) = \quad \quad \quad x_0 =$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} =$$

Answer

$$f(x) = x^3 - x + 3, \quad f'(x) = 3x^2 - 1 \quad x_0 = -2$$

$$\therefore x_1 = -2 - \frac{\{-8 + 2 + 3\}}{11} = -2 + \frac{3}{11} = -1.727$$

Now repeat this process for a second iteration using $x_1 = -1.727$:

Your solution

$$x_2 = x_1 - f(x_1)/f'(x_1) =$$

Answer

$$\begin{aligned} x_2 &= -1.727 - \{- (1.727)^3 + 1.727 + 3\} / \{3(1.727)^2 - 1\} \\ &= -1.727 + \{0.424\} / \{7.948\} = -1.674 \end{aligned}$$

Repeat for a third iteration and state the root to 2 d.p.:

Your solution

$$x_3 = x_2 - f(x_2)/f'(x_2) =$$

Answer

$$\begin{aligned} x_3 &= -1.674 - \{- (1.674)^3 + 1.674 + 3\} / \{3(1.674)^2 - 1\} \\ &= -1.674 + \{0.017\} / \{7.407\} = -1.672 \end{aligned}$$

We conclude the value of the simple root is -1.67 correct to 2 d.p.



Engineering Example 5

Buckling of a strut

The equation governing the buckling load P of a strut with one end fixed and the other end simply supported is given by $\tan \mu L = \mu L$ where $\mu = \sqrt{\frac{P}{EI}}$, L is the length of the strut and EI is the flexural rigidity of the strut. For safe design it is important that the load applied to the strut is less than the lowest buckling load. This equation has no exact solution and we must therefore use the method described in this Workbook to find the lowest buckling load P .

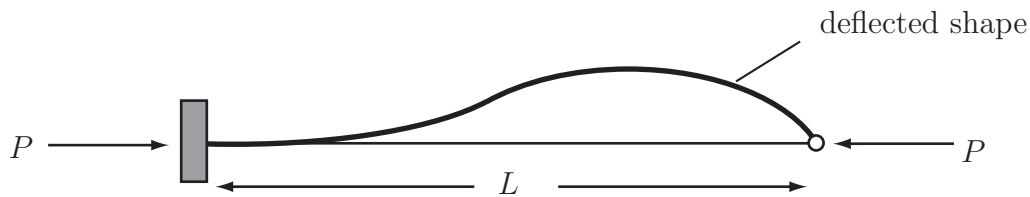
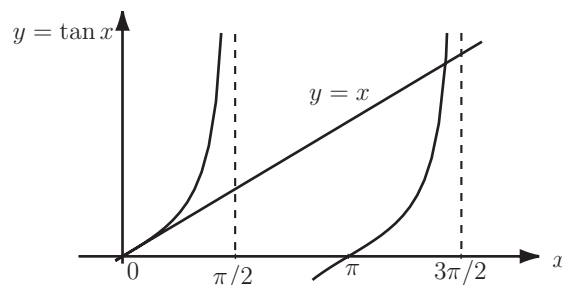


Figure 26

We let $\mu L = x$ and so we need to solve the equation $\tan x = x$. Before starting to apply the Newton-Raphson iteration we must first obtain an approximate solution by plotting graphs of $y = \tan x$ and $y = x$ using the same axes.



From the graph it can be seen that the solution is near to but below $x = 3\pi/2$ (~ 4.7). We therefore start the Newton-Raphson iteration with a value $x_0 = 4.5$.

The equation is rewritten as $\tan x - x = 0$. Let $f(x) = \tan x - x$ then $f'(x) = \sec^2 x - 1 = \tan^2 x$

The Newton-Raphson iteration is $x_{n+1} = x_n - \frac{\tan x_n - x_n}{\tan^2 x_n}$, $x_0 = 4.5$

$$\text{so } x_1 = 4.5 - \frac{\tan(4.5) - 4.5}{\tan^2 4.5} = 4.5 - \frac{0.137332}{21.504847} = 4.493614 \text{ to 7 sig.fig.}$$

Rounding to 4 sig.fig. and iterating:

$$x_2 = 4.494 - \frac{\tan(4.494) - 4.494}{\tan^2 4.494} = 4.494 - \frac{0.004132}{20.229717} = 4.493410 \text{ to 7 sig.fig.}$$

So we conclude that the value of x is 4.493 to 4 sig.fig. As $x = \mu L = \left(\sqrt{P/EI}\right) L$ we find, after re-arrangement, that the smallest buckling load is given by $P = 20.19 \frac{EI}{L^2}$.

Exercises

1. By sketching the function $f(x) = x - 1 - \sin x$ show that there is a simple root near $x = 2$. Use two iterations of the Newton-Raphson method to obtain a better estimate of the root.
2. Obtain an estimation accurate to 2 d.p. of the point of intersection of the curves $y = x - 1$ and $y = \cos x$.

Answers

1. $x_0 = 2, \quad x_1 = 1.936, \quad x_2 = 1.935$

2. The curves intersect when $x - 1 - \cos x = 0$. Solve this using the Newton-Raphson method with initial estimate (say) $x_0 = 1.2$.

The point of intersection is $(1.28342, 0.283437)$ to 6 significant figures.