# Maxima and Minima 12.2

## Introduction

In this Section we analyse curves in the 'local neighbourhood' of a stationary point and, from this analysis, deduce necessary conditions satisfied by local maxima and local minima. Locating the maxima and minima of a function is an important task which arises often in applications of mathematics to problems in engineering and science. It is a task which can often be carried out using only a knowledge of the derivatives of the function concerned. The problem breaks into two parts

- finding the stationary points of the given functions
- distinguishing whether these stationary points are maxima, minima or, exceptionally, points of inflection.

This Section ends with maximum and minimum problems from engineering contexts.

## Prerequisites

Before starting this Section you should ...

- be able to obtain first and second derivatives of simple functions
- be able to find the roots of simple equations

## Learning Outcomes

On completion you should be able to ...

- explain the difference between local and global maxima and minima
- describe how a tangent line changes near a maximum or a minimum
- locate the position of stationary points
- use knowledge of the second derivative to distinguish between maxima and minima

# 1. Maxima and minima

Consider the curve

$$y = f(x) \qquad a \leq x \leq b$$

shown in Figure 7:

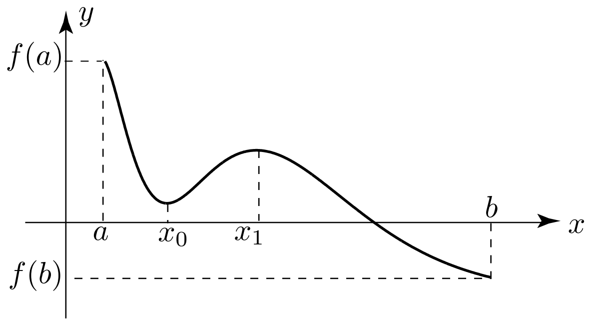



**Figure 7**

By inspection we see that there is no $y$-value greater than that at $x = a$ (i.e. $f(a)$) and there is no value smaller than that at $x = b$ (i.e. $f(b)$). However, the points on the curve at $x_0$ and $x_1$ merit comment. It is clear that in the **near neighbourhood** of $x_0$ all the $y$-values are greater than the $y$-value at $x_0$ and, similarly, in the near neighbourhood of $x_1$ all the $y$-values are less than the $y$-value at $x_1$.

We say $f(x)$ has a **global maximum** at $x = a$ and a **global minimum** at $x = b$ but also has a **local minimum** at $x = x_0$ and a **local maximum** at $x = x_1$.

Our primary purpose in this Section is to see how we might locate the position of the local maxima and the local minima for a smooth function $f(x)$.

A **stationary point** on a curve is one at which the derivative has a zero value. In Figure 8 we have sketched a curve with a maximum and a curve with a minimum.

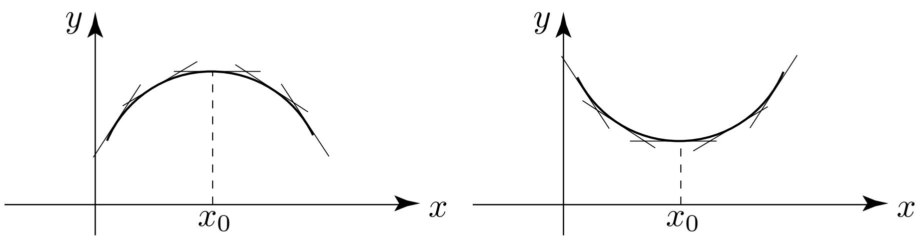



**Figure 8**

By drawing tangent lines to these curves in the near neighbourhood of the local maximum and the local minimum it is obvious that at these points the tangent line is parallel to the $x$-axis so that

$$\left.\frac{df}{dx}\right|_{x_0} = 0$$

## Key Point 3

Points on the curve $y = f(x)$ at which $\frac{df}{dx} = 0$ are called **stationary points** of the function.

However, be careful! A stationary point is not necessarily a local maximum or minimum of the function but may be an exceptional point called a point of inflection, illustrated in Figure 9.

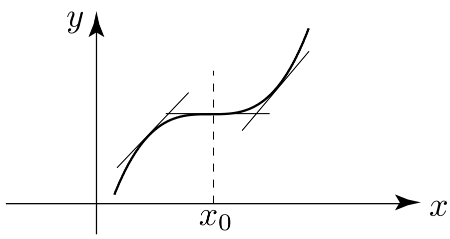



**Figure 9**

## Example 2

Sketch the curve $y = (x-2)^2 + 2$ and locate the stationary points on the curve.

**Solution**

Here $f(x) = (x-2)^2 + 2$ so $\frac{df}{dx} = 2(x-2)$.

At a stationary point $\frac{df}{dx} = 0$ so we have $2(x-2) = 0$ so $x = 2$. We conclude that this function has just one stationary point located at $x = 2$ (where $y = 2$).

By sketching the curve $y = f(x)$ it is clear that this stationary point is a local **minimum**.

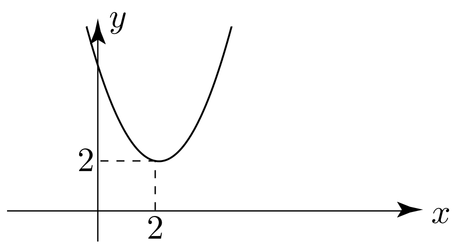



**Figure 10**

**Task**

Locate the position of the stationary points of $f(x) = x^3 - 1.5x^2 - 6x + 10$.

First find $\frac{df}{dx}$:

**Your solution**

$\frac{df}{dx} =$

**Answer**

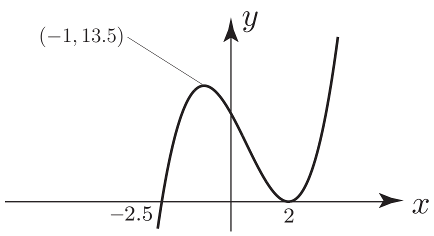
$\frac{df}{dx} = 3x^2 - 3x - 6$

Now locate the stationary points by solving $\frac{df}{dx} = 0$:

**Your solution**

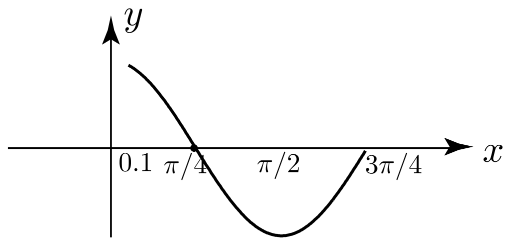
**Answer**

$3x^2 - 3x - 6 = 3(x+1)(x-2) = 0$ so $x = -1$ or $x = 2$. When $x = -1$, $f(x) = 13.5$ and when $x = 2$, $f(x) = 0$, so the stationary points are $(-1, 13.5)$ and $(2, 0)$. We have, in the figure, sketched the curve which confirms our deductions.

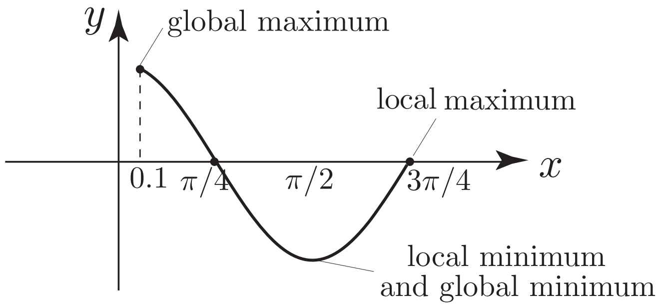



Sketch the curve $y = \cos 2x \qquad 0.1 \leq x \leq \dfrac{3\pi}{4}$ and on it locate the position of the global maximum, global minimum and any local maxima or minima.

**Your solution**

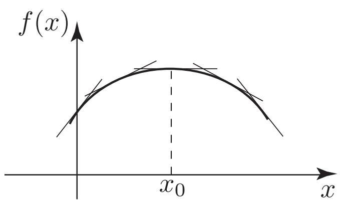



**Answer**




## 2. Distinguishing between local maxima and minima

We might ask if it is possible to predict when a stationary point is a local maximum, a local minimum or a point of inflection without the necessity of drawing the curve. To do this we highlight the general characteristics of curves in the neighbourhood of local maxima and minima.
For example: at a local maximum (located at $x_0$ say) Figure 11 describes the situation:




to the left of the maximum $\dfrac{df}{dx} > 0$

to the right of the maximum $\dfrac{df}{dx} < 0$

**Figure 11**

If we draw a graph of the **derivative** $\dfrac{df}{dx}$ against $x$ then, near a local maximum, it **must** take one of two basic shapes described in Figure 12:

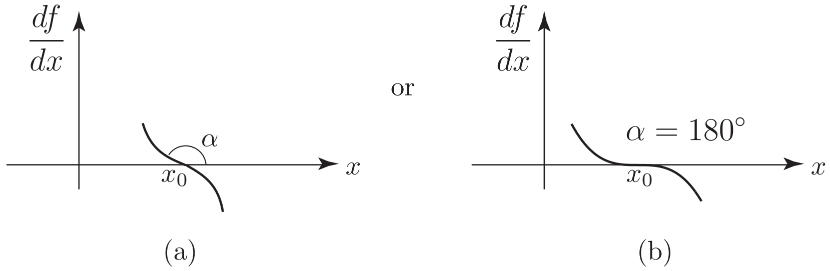
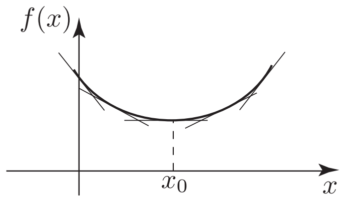



**Figure 12**

In case (a) $\dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\bigg|_{x_0} \equiv \tan\alpha < 0$ whilst in case (b) $\dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\bigg|_{x_0} = 0$

We reach the conclusion that at a stationary point which is a maximum the value of the second derivative $\dfrac{d^2f}{dx^2}$ is either negative or zero.

Near a local minimum the above graphs are inverted. Figure 13 shows a local minimum.




to the left of the minimum $\dfrac{df}{dx} < 0$

to the right of the minimum $\dfrac{df}{dx} > 0$

**Figure 13**

Figure 134 shows the two possible graphs of the derivative:

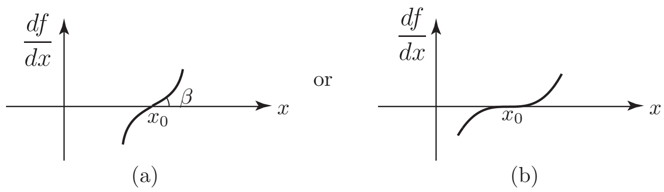
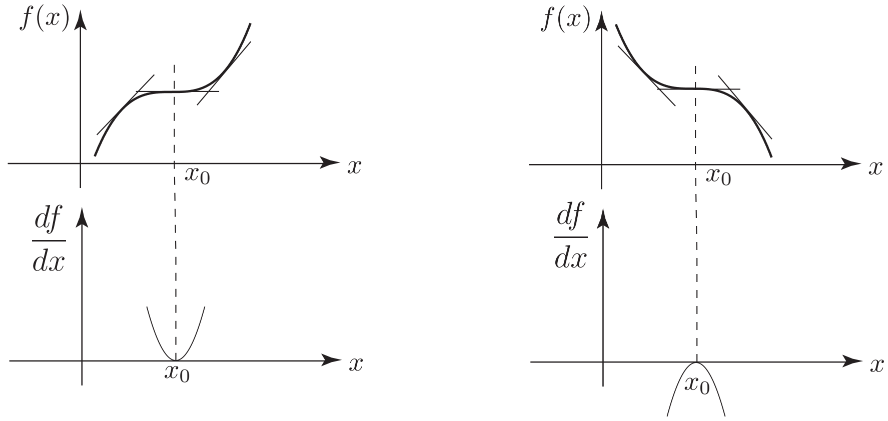



**Figure 14**

Here, for case (a) $\dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\bigg|_{x_0} = \tan\beta > 0$ whilst in (b) $\dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\bigg|_{x_0} = 0.$

In this case we conclude that at a stationary point which is a minimum the value of the second derivative $\dfrac{d^2f}{dx^2}$ is either positive or zero.

For the third possibility for a stationary point - a point of inflection - the graph of $f(x)$ against $x$ and of $\frac{df}{dx}$ against $x$ take one of two forms as shown in Figure 15.




| | | | |
|---|---|---|---|
| to the left of $x_0$ | $\frac{df}{dx} > 0$ | to the left of $x_0$ | $\frac{df}{dx} < 0$ |
| to the right of $x_0$ | $\frac{df}{dx} > 0$ | to the right of $x_0$ | $\frac{df}{dx} < 0$ |

**Figure 15**

For either of these cases $\quad \left.\frac{d}{dx}\left(\frac{df}{dx}\right)\right|_{x_0} = 0$

The sketches and analysis of the shape of a curve $y = f(x)$ in the near neighbourhood of stationary points allow us to make the following important deduction:

## Key Point 4

If $x_0$ locates a stationary point of $f(x)$, so that $\left.\frac{df}{dx}\right|_{x_0} = 0$, then the stationary point

is a local minimum if $\quad \left.\frac{d^2f}{dx^2}\right|_{x_0} > 0$

is a local maximum if $\quad \left.\frac{d^2f}{dx^2}\right|_{x_0} < 0$

is inconclusive if $\quad \left.\frac{d^2f}{dx^2}\right|_{x_0} = 0$

## Example 3

Find the stationary points of the function $f(x) = x^3 - 6x$.

Are these stationary points local maxima or local minima?

**Solution**

$\dfrac{df}{dx} = 3x^2 - 6$. At a stationary point $\dfrac{df}{dx} = 0$ so $3x^2 - 6 = 0$, implying $x = \pm\sqrt{2}$.

Thus $f(x)$ has stationary points at $x = \sqrt{2}$ and $x = -\sqrt{2}$. To decide if these are maxima or minima we examine the value of the second derivative of $f(x)$ at the stationary points.

$\dfrac{d^2f}{dx^2} = 6x$ so $\left.\dfrac{d^2f}{dx^2}\right|_{x=\sqrt{2}} = 6\sqrt{2} > 0$. Hence $x = \sqrt{2}$ locates a local minimum.

Similarly $\left.\dfrac{d^2f}{dx^2}\right|_{x=-\sqrt{2}} = -6\sqrt{2} < 0$. Hence $x = -\sqrt{2}$ locates a local maximum.

A sketch of the curve confirms this analysis:

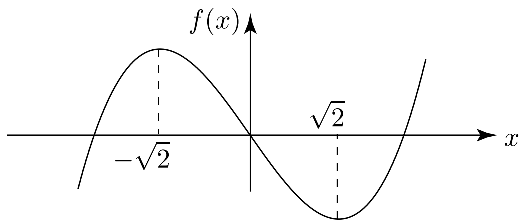



**Figure 16**

## Task

For the function $f(x) = \cos 2x$, $0.1 \leq x \leq 6$, find the positions of any local minima or maxima and distinguish between them.

Calculate the first derivative and locate stationary points:

**Your solution**

$\dfrac{df}{dx} =$

Stationary points are located at:

**Answer**

$\dfrac{df}{dx} = -2\sin 2x.$

Hence stationary points are at values of $x$ in the range specified for which $\sin 2x = 0$ i.e. at $2x = \pi$ or $2x = 2\pi$ or $2x = 3\pi$ (making sure $x$ is within the range $0.1 \leq x \leq 6$)

$\therefore$ Stationary points at $x = \dfrac{\pi}{2},\ x = \pi,\ x = \dfrac{3\pi}{2}$

Now calculate the second derivative:

**Your solution**

$$\frac{d^2f}{dx^2} =$$

**Answer**

$\dfrac{d^2f}{dx^2} = -4\cos 2x$

Finally: evaluate the second derivative at each stationary points and draw appropriate conclusions:

**Your solution**

$$\left.\frac{d^2f}{dx^2}\right|_{x=\frac{\pi}{2}} =$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=\pi} =$$

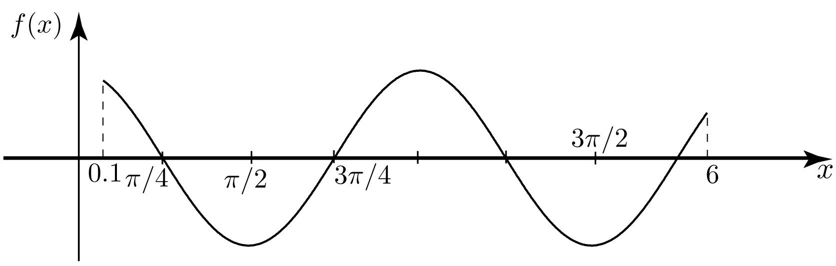
$$\left.\frac{d^2f}{dx^2}\right|_{x=\frac{3\pi}{2}} =$$

**Answer**

$\left.\dfrac{d^2f}{dx^2}\right|_{x=\frac{\pi}{2}} = -4\cos\pi = 4 > 0 \qquad \therefore \qquad x = \dfrac{\pi}{2}$ locates a local minimum.

$\left.\dfrac{d^2f}{dx^2}\right|_{x=\pi} = -4\cos 2\pi = -4 < 0 \qquad \therefore \qquad x = \pi$ locates a local maximum.

$\left.\dfrac{d^2f}{dx^2}\right|_{x=\frac{3\pi}{2}} = -4\cos 3\pi = 4 > 0 \qquad \therefore \qquad x = \dfrac{3\pi}{2}$ locates a local minimum.



**Task**

Determine the local maxima and/or minima of the function $y = x^4 - \frac{1}{3}x^3$

First obtain the positions of the stationary points:

**Your solution**

$f(x) = x^4 - \frac{1}{3}x^3 \qquad \frac{df}{dx} =$

Thus $\frac{df}{dx} = 0$ when:

**Answer**

$\frac{df}{dx} = 4x^3 - x^2 = x^2(4x-1) \qquad \frac{df}{dx} = 0$ when $x = 0$ or when $x = 1/4$

Now obtain the value of the second derivatives at the stationary points:

**Your solution**

$\frac{d^2f}{dx^2} = \qquad \therefore \qquad \left.\frac{d^2f}{dx^2}\right|_{x=0} =$

$\left.\frac{d^2f}{dx^2}\right|_{x=1/4} =$

**Answer**

$\frac{d^2f}{dx^2} = 12x^2 - 2x \qquad \left.\frac{d^2f}{dx^2}\right|_{x=0} = 0,$ which is inconclusive.

$\left.\frac{d^2f}{dx^2}\right|_{x=1/4} = \frac{12}{16} - \frac{1}{2} = \frac{1}{4} > 0$ Hence $x = \frac{1}{4}$ locates a local minimum.

Using this analysis we cannot decide whether the stationary point at $x = 0$ is a local maximum, minimum or a point of inflection. However, just to the left of $x = 0$ the value of $\frac{df}{dx}$ (which equals $x^2(4x-1)$) is negative whilst just to the right of $x = 0$ the value of $\frac{df}{dx}$ is negative again. Hence the stationary point at $x = 0$ is a **point of inflection**. This is confirmed by sketching the curve as in Figure 17.

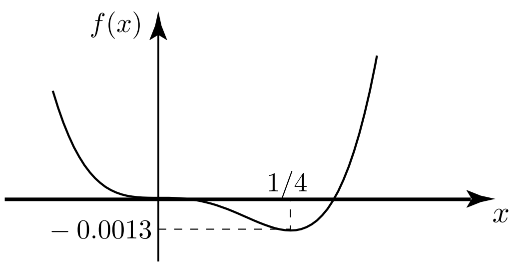
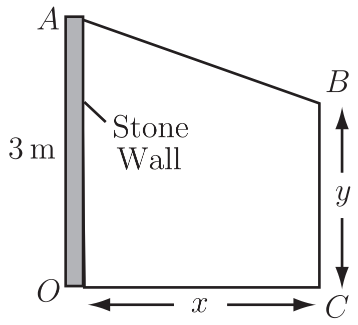



**Figure 17**

A materials store is to be constructed next to a 3 metre high stone wall (shown as $OA$ in the cross section in the diagram). The roof $(AB)$ and front $(BC)$ are to be constructed from corrugated metal sheeting. Only 6 metre length sheets are available. Each of them is to be cut into two parts such that one part is used for the roof and the other is used for the front. Find the dimensions $x, y$ of the store that result in the maximum cross-sectional area. Hence determine the maximum cross-sectional area.




**Your solution**

**Answer**

Note that the store has the shape of a trapezium. So the cross-sectional area $(A)$ of the store is given by the formula: Area = average length of parallel sides × distance between parallel sides:

$$A = \frac{1}{2}(y+3)x \qquad (1)$$

The lengths $x$ and $y$ are related through the fact that $AB + BC = 6$, where $BC = y$ and $AB = \sqrt{x^2+(3-y)^2}$. Hence $\sqrt{x^2+(3-y)^2} + y = 6$. This equation can be rearranged in the following way:

$$\sqrt{x^2+(3-y)^2} = 6-y \quad \Leftrightarrow \quad x^2+(3-y)^2 = (6-y)^2 \text{ i.e. } \quad x^2+9-6y+y^2 = 36-12y+y^2$$

which implies that $\qquad x^2 + 6y = 27 \qquad (2)$

It is necessary to eliminate either $x$ or $y$ from (1) and (2) to obtain an equation in a single variable. Using $y$ instead of $x$ as the variable will avoid having square roots appearing in the expression for the cross-sectional area. Hence from Equation (2)

$$y = \frac{27-x^2}{6} \qquad (3)$$

Substituting for $y$ from Equation (3) into Equation (1) gives

$$A = \frac{1}{2}\left(\frac{27-x^2}{6}+3\right)x = \frac{1}{2}\left(\frac{27-x^2+18}{6}\right)x = \frac{1}{12}\left(45x-x^3\right) \qquad (4)$$

To find turning points, we evaluate $\dfrac{dA}{dx}$ from Equation (4) to get

$$\frac{dA}{dx} = \frac{1}{12}(45-3x^2) \qquad (5)$$

Solving the equation $\dfrac{dA}{dx} = 0$ gives $\dfrac{1}{12}(45-3x^2) = 0 \Rightarrow \qquad 45-3x^2 = 0$

Hence $\quad x = \pm\sqrt{15} = \pm\, 3.8730.$ Only $x > 0$ is of interest, so

$$x = \sqrt{15} = 3.87306 \qquad (6)$$

gives the required turning point.

**Check:** Differentiating Equation (5) and using the positive $x$ solution (6) gives

$$\frac{d^2A}{dx^2} = -\frac{6x}{12} = -\frac{x}{2} = -\frac{3.8730}{2} < 0$$

Since the second derivative is negative then the cross-sectional area is a maximum. This is the only turning point identified for $A > 0$ and it is identified as a maximum. To find the corresponding value of $y$, substitute $x = 3.8730$ into Equation (3) to get $y = \dfrac{27-3.8730^2}{6} = 2.0000$

So the values of $x$ and $y$ that yield the maximum cross-sectional area are 3.8730 m and 2.00000 m respectively. To find the maximum cross-sectional area, substitute for $x = 3.8730$ into Equation (5) to get

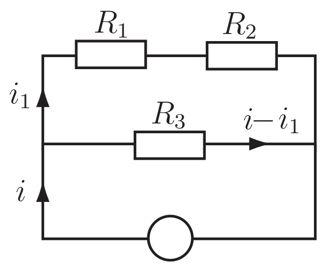
$$A = \frac{1}{2}(45 \times 3.8730 - 3.8730^3) = 9.6825$$

So the maximum cross-sectional area of the store is 9.68 m$^2$ to 2 d.p.

### Equivalent resistance in an electrical circuit

Current distributes itself in the wires of an electrical circuit so as to minimise the total power consumption i.e. the rate at which heat is produced. The power $(p)$ dissipated in an electrical circuit is given by the product of voltage $(v)$ and current $(i)$ flowing in the circuit, i.e. $p = vi$. The voltage across a resistor is the product of current and resistance $(r)$. This means that the power dissipated in a resistor is given by $p = i^2 r$.

Suppose that an electrical circuit contains three resistors $r_1, r_2, r_3$ and $i_1$ represents the current flowing through both $r_1$ and $r_2$, and that $(i - i_1)$ represents the current flowing through $r_3$ (see diagram):




(a) Write down an expression for the power dissipated in the circuit:

**Your solution**

**Answer**

$p = i_1^2 r_1 + i_1^2 r_2 + (i - i_1)^2 r_3$

(b) Show that the power dissipated is a minimum when $i_1 = \dfrac{r_3}{r_1 + r_2 + r_3}\, i$ :

**Your solution**

**Answer**

Differentiate result (a) with respect to $i_1$:

$$\frac{dp}{di_1} = 2i_1r_1 + 2i_1r_2 + 2(i - i_1)(-1)r_3$$

$$= 2i_1(r_1 + r_2 + r_3) - 2ir_3$$

This is zero when

$$i_1 = \frac{r_3}{r_1 + r_2 + r_3}\, i.$$

To check if this represents a minimum, differentiate again:

$$\frac{d^2p}{di_1^2} = 2(r_1 + r_2 + r_3)$$

This is positive, so the previous result represents a minimum.

(c) If $R$ is the equivalent resistance of the circuit, i.e. of $r_1,\ r_2$ and $r_3$, for minimum power dissipation and the corresponding voltage $V$ across the circuit is given by $V = iR = i_1(r_1 + r_2)$, show that

$$R = \frac{(r_1 + r_2)r_3}{r_1 + r_2 + r_3}.$$

**Your solution**

**Answer**

Substituting for $i_1$ in $iR = i_1(r_1 + r_2)$ gives

$$iR = \frac{r_3(r_1 + r_2)}{r_1 + r_2 + r_3}\, i.$$

So

$$R = \frac{(r_1 + r_2)r_3}{r_1 + r_2 + r_3}.$$

**Note** In this problem $R_1$ and $R_2$ could be replaced by a single resistor. However, treating them as separate allows the possibility of considering more general situations (variable resistors or temperature dependent resistors).

## Engineering Example 1

## Water wheel efficiency

### Introduction

A water wheel is constructed with symmetrical curved vanes of angle of curvature $\theta$. Assuming that friction can be taken as negligible, the efficiency, $\eta$, i.e. the ratio of output power to input power, is calculated as

$$\eta = \frac{2(V-v)(1+\cos\theta)v}{V^2}$$

where $V$ is the velocity of the jet of water as it strikes the vane, $v$ is the velocity of the vane in the direction of the jet and $\theta$ is constant. Find the ratio, $v/V$, which gives maximum efficiency and find the maximum efficiency.

### Mathematical statement of the problem

We need to express the efficiency in terms of a single variable so that we can find the maximum value.

$$\text{Efficiency} = \frac{2(V-v)(1+\cos\theta)v}{V^2} = 2\left(1-\frac{v}{V}\right)\frac{v}{V}(1+\cos\theta).$$

Let $\eta =$ Efficiency and $x = \dfrac{v}{V}$ then $\eta = 2x(1-x)(1+\cos\theta)$.

We must find the value of $x$ which maximises $\eta$ and we must find the maximum value of $\eta$. To do this we differentiate $\eta$ with respect to $x$ and solve $\dfrac{d\eta}{dx} = 0$ in order to find the stationary points.

### Mathematical analysis

Now $\eta = 2x(1-x)(1+\cos\theta) = (2x-2x^2)(1+\cos\theta)$

So $\dfrac{d\eta}{dx} = (2-4x)(1+\cos\theta)$

Now $\dfrac{d\eta}{dx} = 0 \Rightarrow 2-4x = 0 \Rightarrow x = \dfrac{1}{2}$ and the value of $\eta$ when $x = \dfrac{1}{2}$ is

$$\eta = 2\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)(1+\cos\theta) = \frac{1}{2}(1+\cos\theta).$$

This is clearly a maximum not a minimum, but to check we calculate $\dfrac{d^2\eta}{dx^2} = -4(1+\cos\theta)$ which is negative which provides confirmation.

### Interpretation

Maximum efficiency occurs when $\dfrac{v}{V} = \dfrac{1}{2}$ and the maximum efficiency is given by

$$\eta = \frac{1}{2}(1+\cos\theta).$$

## Engineering Example 2

### Refraction

#### The problem

A light ray is travelling in a medium $(A)$ at speed $c_A$. The ray encounters an interface with a medium $(B)$ where the velocity of light is $c_B$ . Between two fixed points $P$ in media $A$ and $Q$ in media $B$, find the path through the interface point $O$ that minimizes the time of light travel (see Figure 18). Express the result in terms of the angles of incidence and refraction at the interface and the velocities of light in the two media.

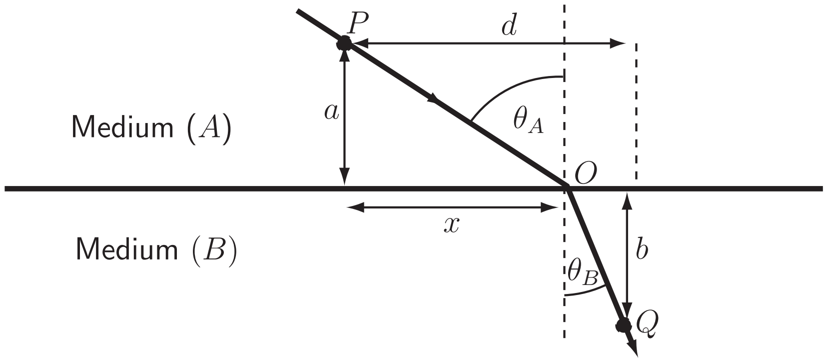



**Figure 18**: Geometry of light rays at an interface

#### The solution

The light ray path is shown as $POQ$ in the above figure where $O$ is a point with variable horizontal position $x$. The points $P$ and $Q$ are fixed and their positions are determined by the constants $a,\ b,\ d$ indicated in the figure. The total path length can be decomposed as $PO + OQ$ so the total time of travel $T(x)$ is given by

$$T(x) = PO/c_A + OQ/c_B. \qquad (1)$$

Expressing the distances $PO$ and $OQ$ in terms of the fixed coordinates $a,\ b,\ d$, and in terms of the unknown position $x$, Equation (1) becomes

$$T(x) = \frac{\sqrt{a^2 + x^2}}{c_A} + \frac{\sqrt{b^2 + (d - x)^2}}{c_B} \qquad (2)$$

It is assumed that the minimum of the travel time is given by the stationary point of $T(x)$ such that

$$\frac{dT}{dx} = 0. \qquad (3)$$

Using the chain rule in (HELM 11.5) to compute (3) given (2) leads to

$$\frac{1}{2}\ \frac{2x}{c_A\sqrt{a^2 + x^2}} + \frac{1}{2}\ \frac{2x - 2d}{c_B\sqrt{b^2 + (d - x)^2}} = 0.$$

After simplification and rearrangement

$$\frac{x}{c_A\sqrt{a^2 + x^2}} = \frac{d - x}{c_B\sqrt{b^2 + (d - x)^2}}.$$

Using the definitions $\sin\theta_A = \dfrac{x}{\sqrt{a^2+x^2}}$ and $\sin\theta_B = \dfrac{d-x}{\sqrt{b^2+(d-x)^2}}$ this can be written as

$$\frac{\sin\theta_A}{c_A} = \frac{\sin\theta_B}{c_B}. \qquad (4)$$

Note that $\theta_A$ and$\theta_B$ are the incidence angles measured from the interface normal as shown in the figure. Equation (4) can be expressed as

$$\frac{\sin\theta_A}{\sin\theta_B} = \frac{c_A}{c_B}$$

which is the well-known law of refraction for geometrical optics and applies to many other kinds of waves. The ratio $\dfrac{c_A}{c_B}$ is a constant called the **refractive index** of medium $(B)$ with respect to medium $(A)$.

## Engineering Example 3

# Fluid power transmission

### Introduction

Power transmitted through fluid-filled pipes is the basis of hydraulic braking systems and other hydraulic control systems. Suppose that power associated with a piston motion at one end of a pipeline is transmitted by a fluid of density $\rho$ moving with positive velocity $V$ along a cylindrical pipeline of constant cross-sectional area $A$. Assuming that the loss of power is mainly attributable to friction and that the friction coefficient $f$ can be taken to be a constant, then the power transmitted, $P$ is given by

$$P = \rho g A(hV - cV^3),$$

where $g$ is the acceleration due to gravity and $h$ is the head which is the height of the fluid above some reference level (= the potential energy per unit weight of the fluid). The constant $c = \dfrac{4fl}{2gd}$ where $l$ is the length of the pipe and $d$ is the diameter of the pipe. The power transmission efficiency is the ratio of power output to power input.

### Problem in words

Assuming that the head of the fluid, $h$, is a constant find the value of the fluid velocity, $V$, which gives a maximum value for the output power $P$. Given that the input power is $P_i = \rho g A V h$, find the maximum power transmission efficiency obtainable.

### Mathematical statement of the problem

We are given that $P = \rho g A(hV - cV^3)$ and we want to find its maximum value and hence maximum efficiency.

To find stationary points for $P$ we solve $\dfrac{dP}{dV} = 0$.

To classify the stationary points we can differentiate again to find the value of $\dfrac{d^2P}{dV^2}$ at each stationary point and if this is negative then we have found a local maximum point. The maximum efficiency is given by the ratio $P/P_i$ at this value of $V$ and where $P_i = \rho g A V h$. Finally we should check that this is the only maximum in the range of $P$ that is of interest.

### Mathematical analysis

$$P = \rho g A(hV - cV^3)$$

$$\frac{dP}{dV} = \rho g A(h - 3cV^2)$$

$$\frac{dP}{dV} = 0 \text{ gives } \rho g A(h - 3cV^2) = 0$$

$$\Rightarrow \quad V^2 = \frac{h}{3c} \quad \Rightarrow \quad V = \pm\sqrt{\frac{h}{3c}} \text{ and as } V \text{ is positive} \quad \Rightarrow \quad V = \sqrt{\frac{h}{3c}}.$$

To show this is a maximum we differentiate $\frac{dP}{dV}$ again giving $\frac{d^2P}{dV^2} = \rho g A(-6cV)$. Clearly this is negative, or zero if $V = 0$. Thus $V = \sqrt{\frac{h}{3c}}$ gives a local maximum value for $P$.

We note that $P = 0$ when $E = \rho g A(hV - cV^3) = 0$, i.e. when $hV - cV^3 = 0$, so $V = 0$ or $V = \sqrt{\frac{h}{C}}$. So the maximum at $V = \sqrt{\frac{h}{3C}}$ is the only max in this range between $0$ and $V = \sqrt{\frac{h}{C}}$.

The efficiency $E$, is given by (input power/output power), so here

$$E = \frac{\rho g A(hV - cV^3)}{\rho g AVh} = 1 - \frac{cV^2}{h}$$

At $V = \sqrt{\frac{h}{3c}}$ then $V^2 = \frac{h}{3c}$ and therefore $E = 1 - \frac{c\frac{h}{3c}}{c} = 1 - \frac{1}{3} = \frac{2}{3}$ or $66\frac{2}{3}\%$.

**Interpretation**

Maximum power transmitted through the fluid when the velocity $V = \sqrt{\frac{h}{3c}}$ and the maximum efficiency is $66\frac{2}{3}$%. Note that this result is independent of the friction and the maximum efficiency is independent of the velocity and (static) pressure in the pipe.

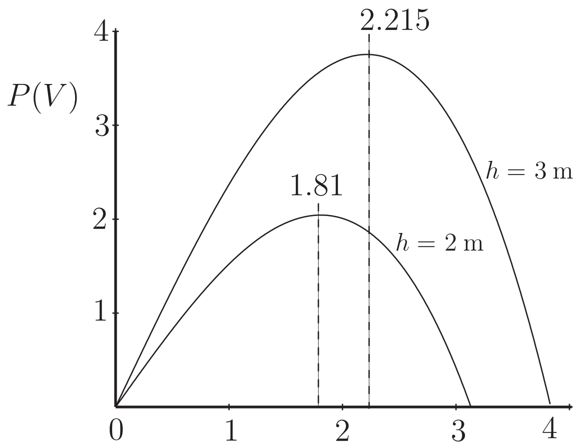



**Figure 19**: Graphs of transmitted power as a function of fluid velocity for two values of the head

Figure 19 shows the maxima in the power transmission for two different values of the head in an oil filled pipe (oil density 1100 kg m$^{-3}$) of inner diameter 0.01 m and coefficient of friction 0.01 and pipe length 1 m.

## Engineering Example 4

# Crank used to drive a piston

### Introduction

A crank is used to drive a piston as in Figure 20.

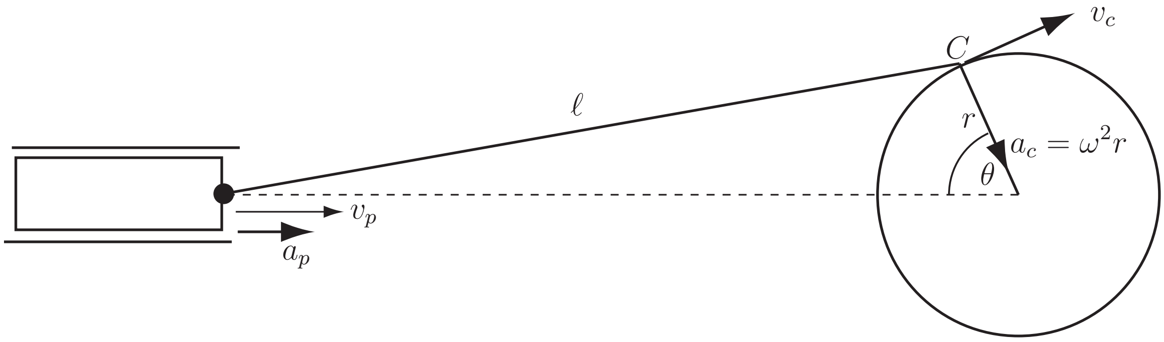



**Figure 20**: Crank used to drive a piston

### Problem

The angular velocity of the crankshaft is the rate of change of the angle $\theta,\ \omega = d\theta/dt$. The piston moves horizontally with velocity $v_p$ and acceleration $a_p$; $r$ is the length of the crank and $l$ is the length of the connecting rod. The crankpin performs circular motion with a velocity of $v_c$ and centripetal acceleration of $\omega^2 r$. The acceleration $a_p$ of the piston varies with $\theta$ and is related by

$$a_p = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{l}\right)$$

Find the maximum and minimum values of the acceleration $a_p$ when $r = 150$ mm and $l = 375$ mm.

### Mathematical statement of the problem

We need to find the stationary values of $a_p = \omega^2 r\left(\cos\theta + \dfrac{r\cos 2\theta}{l}\right)$ when $r = 150$ mm and $l = 375$ mm. We do this by solving $\dfrac{da_p}{d\theta} = 0$ and then analysing the stationary points to decide whether they are a maximum, minimum or point of inflexion.

### Mathematical analysis.

$$a_p = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{l}\right) \text{ so } \frac{da_p}{d\theta} = \omega^2 r\left(-\sin\theta - \frac{2r\sin 2\theta}{l}\right).$$

To find the maximum and minimum acceleration we need to solve

$$\frac{da_p}{d\theta} = 0 \Leftrightarrow \omega^2 r\left(-\sin\theta - \frac{2r\sin 2\theta}{l}\right) = 0.$$

$$\sin\theta + \frac{2r}{l}\sin 2\theta = 0 \Leftrightarrow \sin\theta + \frac{4r}{l}\sin\theta\cos\theta = 0$$

$$\Leftrightarrow \quad \sin\theta\left(1+\frac{4r}{l}\cos\theta\right)=0$$

$$\Leftrightarrow \quad \sin\theta=0 \text{ or } \cos\theta=-\frac{l}{4r} \text{ and as } r=150 \text{ mm and } l=375 \text{ mm}$$

$$\Leftrightarrow \quad \sin\theta=0 \text{ or } \cos\theta=-\frac{5}{8}$$

**CASE 1: $\sin\theta = 0$**

If $\sin\theta=0$ then $\theta=0$ or $\theta=\pi$. If $\theta=0$ then $\cos\theta=\cos 2\theta=1$

so $\quad a_p=\omega^2 r\left(\cos\theta+\frac{r\cos 2\theta}{l}\right)=\omega^2 r\left(1+\frac{r}{l}\right)=\omega^2 r\left(1+\frac{2}{5}\right)=\frac{7}{5}\omega^2 r$

If $\theta=\pi$ then $\cos\theta=-1,\ \cos 2\theta=1$ so

$$a_p=\omega^2 r\left(\cos\theta+\frac{r\cos 2\theta}{l}\right)=\omega^2 r\left(-1+\frac{r}{l}\right)=\omega^2 r\left(-1+\frac{2}{5}\right)=-\frac{3}{5}\omega^2 r$$

In order to classify the stationary points, we differentiate $\frac{da_p}{d\theta}$ with respect to $\theta$ to find the second derivative:

$$\frac{d^2a_p}{d\theta^2}=\omega^2 r\left(-\cos\theta-\frac{4r\cos 2\theta}{l}\right)=-\omega^2 r\left(\cos\theta+\frac{4r\cos 2\theta}{l}\right)$$

At $\theta=0$ we get $\frac{d^2a_p}{d\theta^2}=-\omega^2 r\left(1+\frac{4r}{l}\right)$ which is negative.

So $\theta=0$ gives a **maximum** value and $a_p=\frac{7}{5}\omega^2 r$ is the value at the maximum.

At $\theta=\pi$ we get $\frac{d^2a_p}{d\theta^2}=-\omega^2 r\left(-1+\frac{4}{l}\right)=-\omega^2 r\left(\frac{3}{5}\right)$ which is negative.

So $\theta=\pi$ gives a **maximum** value and $a_p=-\frac{3}{5}\omega^2 r$

**CASE 2: $\cos\theta = -\frac{5}{8}$**

If $\cos\theta=-\frac{5}{8}$ then $\cos 2\theta=2\cos^2\theta-1=2\left(\frac{5}{8}\right)^2-1$ so $\cos 2\theta=-\frac{7}{32}$.

$$a_p=\omega^2 r\left(\cos\theta+\frac{r\cos 2\theta}{l}\right)=\omega^2 r\left(-\frac{5}{8}+-\frac{7}{32}\times\frac{2}{5}\right)=\frac{57}{80}\omega^2 r.$$

At $\cos\theta=-\frac{5}{8}$ we get $\frac{d^2a_p}{d\theta^2}=\omega^2 r\left(-\cos\theta-\frac{4r\cos 2\theta}{l}\right)=\omega^2 r\left(\frac{5}{8}+\frac{4r}{l}\,\frac{7}{32}\right)$ which is positive.

So $\cos\theta=-\frac{5}{8}$ gives a **minimum** value and $a_p=-\frac{57}{80}\omega^2 r$

Thus the values of $a_p$ at the stationary points are:-

$\frac{7}{5}\omega^2 r$ (maximum), $-\frac{3}{5}\omega^2 r$ (maximum) and $-\frac{57}{80}\omega^2 r$ (minimum).

So the overall maximum value is $1.4\omega^2 r = 0.21\omega^2$ and the minimum value is $-0.7125\omega^2 r = -0.106875\omega^2$ where we have substituted $r = 150$ mm (= 0.15 m) and $l = 375$ mm ($= 0.375$ m).

**Interpretation**

The maximum acceleration occurs when $\theta = 0$ and $a_p = 0.21\omega^2$.

The minimum acceleration occurs when $\cos\theta = -\dfrac{5}{8}$ and $a_p = -0.106875\omega^2$.

## Exercises

1. Locate the stationary points of the following functions and distinguish among them as maxima, minima and points of inflection.

   (a) $f(x) = x - \ln|x|$. [Remember $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$]

   (b) $f(x) = x^3$

   (c) $f(x) = \dfrac{(x-1)}{(x+1)(x-2)} \qquad -1 < x < 2$

2. A perturbation in the temperature of a stream leaving a chemical reactor follows a decaying sinusoidal variation, according to

   $$T(t) = 5\text{exp}(-at)\sin(\omega t)$$

   where $a$ and $\omega$ are positive constants.

   (a) Sketch the variation of temperature with time.

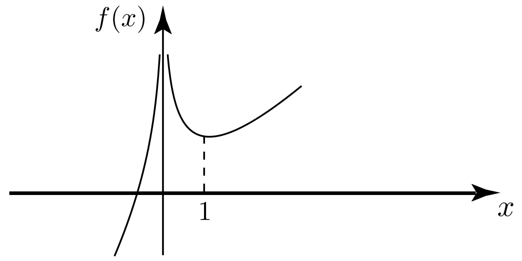
   (b) By examining the behaviour of $\dfrac{dT}{dt}$, show that the maximum temperatures occur at times of $\left(\tan^{-1}(\dfrac{\omega}{a}) + 2\pi n\right)/\omega$.

**Answers**

1. (a) $\dfrac{df}{dx} = 1 - \dfrac{1}{x} = 0$ when $x = 1$

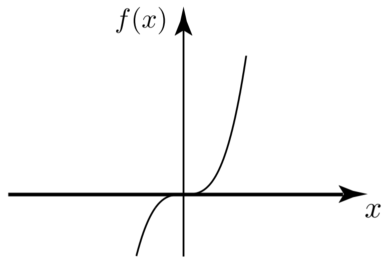
$\dfrac{d^2f}{dx^2} = \dfrac{1}{x^2} \qquad \left.\dfrac{d^2f}{dx^2}\right|_{x=1} = 1 > 0$

$\therefore \quad x = 1, y = 1$ locates a local minimum.




(b) $\dfrac{df}{dx} = 3x^2 = 0$ when $x = 0 \qquad \dfrac{d^2f}{dx^2} = 6x = 0$ when $x = 0$

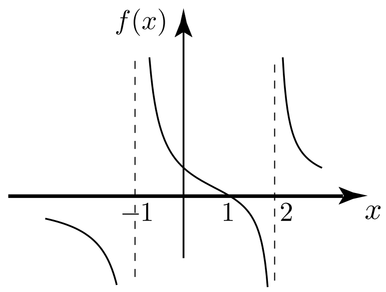
However, $\dfrac{df}{dx} > 0$ on either side of $x = 0$ so $(0, 0)$ is a point of inflection.




(c) $\dfrac{df}{dx} = \dfrac{(x+1)(x-2) - (x-1)(2x-1)}{(x+1)(x-2)}$

This is zero when $(x+1)(x-2) - (x-1)(2x-1) = 0 \quad$ i.e. $x^2 - 2x + 3 = 0$

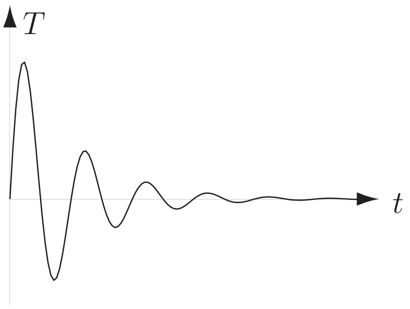
However, this equation has no real roots (since $b^2 < 4ac$) and so $f(x)$ has no stationary points. The graph of this function confirms this:




Nevertheless $f(x)$ **does** have a point of inflection at $x = 1, y = 0$ as the graph shows, although at that point $\dfrac{dy}{dx} \neq 0$.

**Answer**

2. (a)




(b) $\dfrac{dT}{dt} = 0$ implies $\tan \omega t = \dfrac{\omega}{a}$, so $\tan \omega t > 0$ and

$$\omega t = \tan^{-1}\left(\frac{\omega}{a}\right) + k\pi, \qquad k \text{ integer}$$

Examination of $\dfrac{d^2T}{dt^2}$ reveals that only even values of $k$ give $\dfrac{d^2T}{dt^2} < 0$ for a maximum so setting $k = 2n$ gives the required answer.