The Exponential Form of a Complex Number 10.3

Introduction

In this Section we introduce a third way of expressing a complex number: the exponential form. We shall discover, through the use of the complex number notation, the intimate connection between the exponential function and the trigonometric functions. We shall also see, using the exponential form, that certain calculations, particularly multiplication and division of complex numbers, are even easier than when expressed in polar form.

The exponential form of a complex number is in widespread use in engineering and science.

Prerequisites

Before starting this Section you should ...

- be able to convert from degrees to radians
- understand how to use the Cartesian and polar forms of a complex number
- be familiar with the hyperbolic functions $\cosh x$ and $\sinh x$

Learning Outcomes

On completion you should be able to ...

- explain the relations between the exponential function e^x and the trigonometric functions $\cos x$, $\sin x$
- interchange between Cartesian, polar and exponential forms of a complex number
- explain the relation between hyperbolic and trigonometric functions

1. Series expansions for exponential and trigonometric functions

We have, so far, considered two ways of representing a complex number:

$z = a + \mathrm{i}b$ Cartesian form

or

$z = r(\cos\theta + \mathrm{i}\sin\theta)$ polar form

In this Section we introduce a third way of denoting a complex number: the **exponential form**.

If x is a real number then, as we shall verify in HELM 16, the exponential number e raised to the power x can be written as a series of powers of x:

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

in which $n! = n(n-1)(n-2)\ldots(3)(2)(1)$ is the factorial of the integer n. Although there are an infinite number of terms on the right-hand side, in any practical calculation we could only use a finite number. For example if we choose $x = 1$ (and taking only six terms) then

$$\begin{aligned}\mathrm{e}^1 &\approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}\\ &= 2 + 0.5 + 0.16666 + 0.04166 + 0.00833\\ &= 2.71666\end{aligned}$$

which is fairly close to the accurate value of $\mathrm{e} = 2.71828$ (to 5 d.p.)

We ask you to accept that e^x, for any real value of x, is the **same as** $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and that if we wish to calculate e^x for a *particular value* of x we will only take a finite number of terms in the series. Obviously the more terms we take in any particular calculation the more accurate will be our calculation.

As we shall also see in HELM 16, similar series expansions exist for the trigonometric functions $\sin x$ and $\cos x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

in which x is measured in radians.

The observant reader will see that these two series for $\sin x$ and $\cos x$ are similar to the series for e^x. Through the use of the symbol i (where $\mathrm{i}^2 = -1$) we will examine this close correspondence.

In the series for e^x replace x on both left-hand and right-hand sides by $\mathrm{i}\theta$ to give:

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + (\mathrm{i}\theta) + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \cdots$$

Then, as usual, replace every occurrence of i^2 by -1 to give

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \mathrm{i}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \mathrm{i}\frac{\theta^5}{5!} + \cdots$$

which, when re-organised into real and imaginary terms gives, finally:

$$\begin{aligned} e^{i\theta} &= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Key Point 8

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

Example 5

Find complex number expressions, in Cartesian form, for

(a) $e^{i\pi/4}$ (b) e^{-i} (c) $e^{i\pi}$

We use Key Point 8:

Solution

(a) $e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

(b) $e^{-i} = \cos(-1) + i\sin(-1) = 0.540 - i(0.841)$ don't forget: use **radians**

(c) $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$

2. The exponential form

Since $z = r(\cos\theta + i\sin\theta)$ and since $e^{i\theta} = \cos\theta + i\sin\theta$ we therefore obtain another way in which to denote a complex number: $z = re^{i\theta}$, called the **exponential form**.

Key Point 9

The exponential form of a complex number is

$$z = re^{i\theta} \quad \text{in which} \quad r = |z| \quad \text{and} \quad \theta = \arg(z)$$

so

$$z = re^{i\theta} = r(\cos\theta + i\ \sin\theta)$$

Task

Express $z = 3\mathrm{e}^{\mathrm{i}\pi/6}$ in Cartesian form, correct to 2 d.p.

Use Key Point 9:

Your solution

Answer

$$\begin{aligned} z = 3\mathrm{e}^{\mathrm{i}\pi/6} &= 3(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}) \\ &= 3(0.8660 + \mathrm{i}0.5000) \\ &= 2.60 + 1.50\mathrm{i} \qquad \text{to 2 d.p.} \end{aligned}$$

Example 6

If $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $w = t\mathrm{e}^{\mathrm{i}\phi}$ then find expressions for (a) z^{-1} (b) z^* (c) zw

Solution

(a) If $z = r\mathrm{e}^{\mathrm{i}\theta}$ then $z^{-1} = \dfrac{1}{r\mathrm{e}^{\mathrm{i}\theta}} = \dfrac{1}{r}\mathrm{e}^{-\mathrm{i}\theta}$ using the normal rules for indices.

(b) Working in polar form: if $z = r\mathrm{e}^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$ then

$$z^* = r(\cos\theta - \mathrm{i}\sin\theta) = r(\cos(-\theta) + \mathrm{i}\sin(-\theta)) = r\mathrm{e}^{-\mathrm{i}\theta}$$

since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. In fact this reflects the general rule: to find the complex conjugate of any expression simply replace i by $-$i wherever it occurs in the expression.

(c) $zw = (r\mathrm{e}^{\mathrm{i}\theta})(t\mathrm{e}^{\mathrm{i}\phi}) = rt\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\phi} = rt\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\phi} = rt\mathrm{e}^{\mathrm{i}(\theta+\phi)}$ which is again the result we are familiar with: when complex numbers are multiplied their moduli multiply and their arguments add.

We see that in some circumstances the exponential form is even more convenient than the polar form since we need not worry about cumbersome trigonometric relations.

Task

Express the following complex numbers in exponential form:

(a) $z = 1 - \mathsf{i}$ (b) $z = 2 + 3\mathsf{i}$ (c) $z = -6$.

Your solution

(a)

Answer

$z = \sqrt{2}\mathsf{e}^{\mathsf{i}7\pi/4}$ (or, equivalently, $\sqrt{2}\mathsf{e}^{-\mathsf{i}\pi/4}$)

Your solution

(b)

Answer

$z = \sqrt{13}\mathsf{e}^{\mathsf{i}(0.9828)}$

Your solution

(c)

Answer

$z = 6\mathsf{e}^{\mathsf{i}\pi}$

3. Hyperbolic and trigonometric functions

We have seen in subsection 1 (Key Point 8) that

$$\mathsf{e}^{\mathsf{i}\theta} = \cos\theta + \mathsf{i}\sin\theta$$

It follows from this that

$$\mathsf{e}^{-\mathsf{i}\theta} = \cos(-\theta) + \mathsf{i}\sin(-\theta) = \cos\theta - \mathsf{i}\sin\theta$$

Now if we add these two relations together we obtain

$$\cos\theta = \frac{\mathsf{e}^{\mathsf{i}\theta} + \mathsf{e}^{-\mathsf{i}\theta}}{2}$$

whereas if we subtract the second from the first we have

$$\sin\theta = \frac{\mathsf{e}^{\mathsf{i}\theta} - \mathsf{e}^{-\mathsf{i}\theta}}{2\mathsf{i}}$$

These new relations are reminiscent of the hyperbolic functions introduced in HELM 6. There we defined $\cosh x$ and $\sinh x$ in terms of the exponential function:

$$\cosh x = \frac{\mathsf{e}^{x} + \mathsf{e}^{-x}}{2} \qquad \sinh x = \frac{\mathsf{e}^{x} - \mathsf{e}^{-x}}{2}$$

In fact, if we replace x by $\mathsf{i}\theta$ in these last two equations we obtain

$$\cosh(\mathrm{i}\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2} \equiv \cos\theta \qquad \text{and} \qquad \sinh(\mathrm{i}\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2} \equiv \mathrm{i}\sin\theta$$

Although, by our notation, we have implied that both x and θ are real quantities in fact these expressions for $\cosh$ and $\sinh$ in terms of $\cos$ and $\sin$ are quite general.

Key Point 10

If z is *any* complex number then

$$\cosh(\mathrm{i}z) \equiv \cos z \qquad \text{and} \qquad \sinh(\mathrm{i}z) \equiv \mathrm{i}\sin z$$

Equivalently, replacing z by $\mathrm{i}z$:

$$\cosh z \equiv \cos(\mathrm{i}z) \qquad \text{and} \qquad \mathrm{i}\sinh z \equiv \sin(\mathrm{i}z)$$

Task

Given that $\cos^2 z + \sin^2 z \equiv 1$ for all z then, utilising complex numbers, obtain the equivalent identity for hyperbolic functions.

Your solution

Answer

You should obtain $\cosh^2 z - \sinh^2 z \equiv 1$ since, if we replace z by iz in the given identity then $\cos^2(\mathrm{i}z) + \sin^2(\mathrm{i}z) \equiv 1$. But as noted above $\cos(\mathrm{i}z) \equiv \cosh z$ and $\sin(\mathrm{i}z) \equiv \mathrm{i}\sinh z$ so the result follows.

Further analysis similar to that in the above task leads to **Osborne's rule**:

Key Point 11

Osborne's Rule

Hyperbolic function identities are obtained from trigonometric function identities by replacing $\sin\theta$ by $\sinh\theta$ and $\cos\theta$ by $\cosh\theta$ except that every occurrence of $\sin^2\theta$ is replaced by $-\sinh^2\theta$.

Example 7

Use Osborne's rule to obtain the hyperbolic identity equivalent to
$1 + \tan^2\theta \equiv \sec^2\theta$.

Solution

Here $1 + \tan^2\theta \equiv \sec^2\theta$ is equivalent to $1 + \dfrac{\sin^2\theta}{\cos^2\theta} \equiv \dfrac{1}{\cos^2\theta}$. Hence if

$$\sin^2\theta \to -\sinh^2\theta \qquad \text{and} \qquad \cos^2\theta \to \cosh^2\theta$$

then we obtain

$$1 - \frac{\sinh^2\theta}{\cosh^2\theta} \equiv \frac{1}{\cosh^2\theta} \qquad \text{or, equivalently,} \qquad 1 - \tanh^2\theta \equiv \text{sech}^2\theta$$

Engineering Example 1

Feedback applied to an amplifier

Feedback is applied to an amplifier such that

$$A' = \frac{A}{1 - \beta A}$$

where A', A and β are complex quantities. A is the amplifier gain, A' is the gain with feedback and β is the proportion of the output which has been fed back.

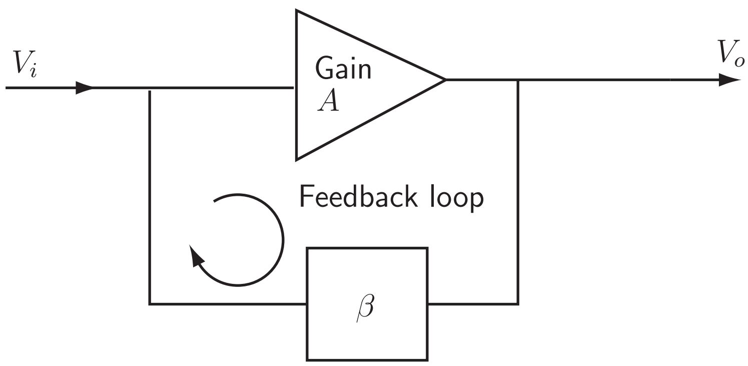

Figure 8: An amplifier with feedback

(a) If at 30 Hz, $A = -500$ and $\beta = 0.005e^{8\pi i/9}$, calculate A' in exponential form.

(b) At a particular frequency it is desired to have $A' = 300e^{5\pi i/9}$ where it is known that $A = 400e^{11\pi i/18}$. Find the value of β necessary to achieve this gain modification.

Mathematical statement of the problem

For (a): substitute $A = -500$ and $\beta = 0.005e^{8\pi i/9}$ into $A' = \dfrac{A}{1-\beta A}$ in order to find A'.

For (b): we need to solve for β when $A' = 300e^{5\pi i/9}$ and $A = 400e^{11\pi i/18}$.

Mathematical analysis

(a) $$A' = \frac{A}{1-\beta A} = \frac{-500}{1 - 0.005e^{8\pi i/9} \times (-500)} = \frac{-500}{1 + 2.5e^{8\pi i/9}}$$

Expressing the bottom line of this expression in Cartesian form this becomes:

$$A' = \frac{-500}{1 + 2.5\cos(\frac{8\pi}{9}) + 2.5i\sin(\frac{8\pi}{9})} \approx \frac{-500}{-1.349 + 0.855i}$$

Expressing both the top and bottom lines in exponential form we get:

$$A' \approx \frac{500e^{i\pi}}{1.597e^{i2.576}} \approx 313e^{0.566i}$$

(b) $$A' = \frac{A}{1-\beta A} \to A'(1-\beta A) = A \to -\beta AA' = A - A'$$

i.e. $$\beta = \frac{A' - A}{AA'} \to \beta = \frac{1}{A} - \frac{1}{A'}$$

So

$$\beta = \frac{1}{A} - \frac{1}{A'} = \frac{1}{400e^{11\pi i/18}} - \frac{1}{300e^{5\pi i/9}} \approx 0.0025e^{-11\pi i 18} - 0.00333e^{-5\pi i/9}$$

Expressing both complex numbers in Cartesian form gives

$$\beta = 0.0025\cos(-\frac{11\pi}{18}) + 0.0025i\sin(-\frac{11\pi}{18}) - 0.00333\cos(-\frac{5\pi}{9}) - 0.00333i\sin(-\frac{5\pi}{9})$$
$$= -2.768 \times 10^{-4} + 9.3017 \times 10^{-4}i = 9.7048 \times 10^{-4}e^{1.86i}$$

So to 3 significant figures $\beta = 9.70 \times 10^{-4}e^{1.86i}$

Exercises

1. Two standard identities in trigonometry are $\sin 2z \equiv 2\sin z \cos z$ and $\cos 2z \equiv \cos^2 z - \sin^2 z$. Use Osborne's rule to obtain the corresponding identities for hyperbolic functions.

2. Express $\sinh(a + \mathrm{i}b)$ in Cartesian form.

3. Express the following complex numbers in Cartesian form (a) $3\mathrm{e}^{\mathrm{i}\pi/3}$ (b) $\mathrm{e}^{-2\pi\mathrm{i}}$ (c) $\mathrm{e}^{\mathrm{i}\pi/2}\mathrm{e}^{\mathrm{i}\pi/4}$.

4. Express the following complex numbers in exponential form

   (a) $z = 2 - \mathrm{i}$ (b) $z = 4 - 3\mathrm{i}$ (c) z^{-1} where $z = 2 - 3\mathrm{i}$.

5. Obtain the real and imaginary parts of $\sinh(1 + \dfrac{\mathrm{i}\pi}{6})$.

Answers

1. $\sinh 2z \equiv 2\sinh z \cosh z$, $\cosh 2z \equiv \cosh^2 z + \sinh^2 z$.

2. $$\begin{aligned}\sinh(a + \mathrm{i}b) &\equiv \sinh a \cosh \mathrm{i}b + \cosh a \sinh \mathrm{i}b \\ &\equiv \sinh a \cos b + \cosh a(\mathrm{i}\sin b) \\ &\equiv \sinh a \cos b + \mathrm{i}\cosh a \sin b\end{aligned}$$

3. (a) $1.5 + \mathrm{i}(2.598)$ (b) 1 (c) $-0.707 + \mathrm{i}(0.707)$

4. (a) $\sqrt{5}\mathrm{e}^{\mathrm{i}(5.820)}$ (b) $5\mathrm{e}^{\mathrm{i}(5.6397)}$ (c) $2 - 3\mathrm{i} = \sqrt{13}\mathrm{e}^{\mathrm{i}(5.300)}$ therefore $\dfrac{1}{2 - 3\mathrm{i}} = \dfrac{1}{\sqrt{13}}\mathrm{e}^{-\mathrm{i}(5.300)}$

5. $\sinh(1 + \dfrac{\mathrm{i}\pi}{6}) = \dfrac{\sqrt{3}}{2}\sinh 1 + \dfrac{\mathrm{i}}{2}\cosh 1 = 1.0178 + \mathrm{i}(0.7715)$