Contents 10

Complex Numbers

Learning outcomes

In this Workbook you will learn what a complex number is and how to combine complex numbers together using the familiar operations of addition, subtraction, multiplication and division. You will also learn how to describe a complex number graphically using the Argand diagram. The connection between the exponential function and the trigonometric functions is explained. You will understand how De Moivre's theorem is used to obtain fractional powers of complex numbers.

Complex Arithmetic 10.1

Introduction

Complex numbers are used in many areas of engineering and science. In this Section we define what a complex number is and explore how two such numbers may be combined together by adding, subtracting, multiplying and dividing. We also show how to find 'complex roots' of polynomial equations.

A **complex number** is a generalisation of an ordinary real number. In fact, as we shall see, a complex number is a pair of real numbers ordered in a particular way. Fundamental to the study of complex numbers is the symbol i with the strange looking property $\mathrm{i}^2 = -1$. Apart from this property complex numbers follow the usual rules of number algebra.

Prerequisites

Before starting this Section you should . . .

- be able to add, subtract, multiply and divide real numbers
- be able to combine algebraic fractions together
- understand what a polynomial is
- have a knowledge of trigonometric identities

Learning Outcomes

On completion you should be able to . . .

- combine complex numbers together
- find the modulus and conjugate of a complex number
- obtain complex solutions to polynomial equations

1. What is a complex number?

We assume that you are familiar with the properties of ordinary numbers; examples are

$$1,\ -2,\ \frac{3}{10},\ 2.634,\ -3.111,\ \pi,\ e,\ \sqrt{2}$$

We all know how to add, subtract, multiply and divide such numbers. We are aware that the numbers can be positive or negative or zero and also aware of their geometrical interpretation as being represented by points on a 'real' axis known as a number line (Figure 1).

Figure 1

The real axis is a line with a direction (usually chosen to be from left to right) indicated by an arrow. We shall refer to this as the x-axis. On this axis we select a point, arbitrarily, and refer to this as the **origin** O. The origin (where zero is located) distinguishes positive numbers from negative numbers:

- to the right of the origin are the positive numbers
- to the left of the origin are the negative numbers

Thus we can 'locate' the numbers in our example. See Figure 2.

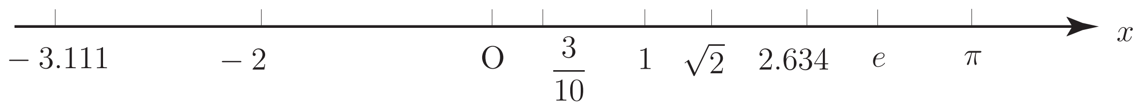

Figure 2

From now on we shall refer to these 'ordinary' numbers as **real** numbers. We can formalise the algebra of real numbers into a set of rules which they obey.
So if x_1, x_2 and x_3 are any three real numbers then we know that, in particular:

1. $x_1 + x_2 = x_2 + x_1 \qquad x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$
2. $1 \times x_1 = x_1 \qquad 0 \times x_1 = 0$
3. $x_1 \times x_2 = x_2 \times x_1 \qquad x_1 \times (x_2 + x_3) = x_1 \times x_2 + x_1 \times x_3$

Also, in multiplication we are familiar with the elementary rules:

(positive) $\times$ (positive) = positive (positive) $\times$ (negative) = negative

(negative) $\times$ (positive) = negative (negative) $\times$ (negative) = positive

It follows that if x represents *any* real number then

$$x^2 \geq 0$$

in words, **the square of a real number is always non-negative**.

In this Workbook we will consider a kind of number (a generalisation of a real number) whose square is not necessarily positive (and not necessarily real either). Don't worry that i 'does not exist'. Because of that it is called imaginary! We just define it and get on and use it and it then turns out to be very useful and important in many practical applications. However, it is important to get to know how to handle complex numbers before using them in calculations. This will not be difficult as the new set of rules is, in fact, precisely the same set of rules obeyed by the 'real' numbers. These new numbers are called complex numbers.

A **complex number** is an ordered pair of real numbers, usually denoted by z or w etc. So if a, b are real numbers then we designate a complex number through:

$$z = a + \mathrm{i}b$$

where i is a symbol obeying the rule

$$\mathrm{i}^2 = -1$$

For simplicity we shall assume we can write

$$\mathrm{i} = \sqrt{-1}.$$

(Often, particularly in engineering applications, the symbol j is used instead of i). Also note that, conventially, examples of actual complex numbers such as $2 + 3\mathrm{i}$ are written like this and not $2 + \mathrm{i}3$. Again we ask the reader to accept matters at this stage without worrying about the meaning of finding the square root of a negative number. Using this notation we can write

$$\sqrt{-4} = \sqrt{(4)(-1)} = \sqrt{4}\sqrt{-1} = 2\mathrm{i} \text{ etc.}$$

Key Point 1

The symbol i is such that

$$\mathrm{i}^2 = -1$$

Using the normal rules of algebra it follows that

$$\mathrm{i}^3 = \mathrm{i}^2 \times \mathrm{i} = -\mathrm{i} \qquad \mathrm{i}^4 = \mathrm{i}^2 \times \mathrm{i}^2 = (-1) \times (-1) = 1$$

and so on.

Simple examples of complex numbers are

$$z_1 = 3 + 2\mathrm{i} \qquad z_2 = -3 + (2.461)\mathrm{i} \qquad z_3 = 17\mathrm{i} \qquad z_4 = 3 + 0\mathrm{i} = 3$$

Generally, if $z = a + \mathrm{i}b$ then 'a' is called the **real part** of z, or $\mathrm{Re}(z)$ for short, and 'b' is called the **imaginary part** of z or $\mathrm{Im}(z)$. The fourth example indicates that the real numbers can be considered a subset of the complex numbers.

Key Point 2

If $\quad z = a + \mathrm{i}b \quad$ then $\quad \mathrm{Re}(z) = a \quad$ and $\quad \mathrm{Im}(z) = b$

Both the real and imaginary parts of a complex number are **real**.

Key Point 3

Two complex numbers $z = a + ib$ and $w = c + id$ are said to be **equal** if and only if both their real parts are the same and both their imaginary parts are the same, that is

$$a = c \qquad \text{and} \qquad b = d$$

Key Point 4

The **modulus** of a complex number $z = a + ib$ is denoted by $|z|$ and is defined by

$$|z| = \sqrt{a^2 + b^2}$$

so that the modulus is always a non-negative real number.

Example 1

If $z = 3 - 2\text{i}$ then find $\text{Re}(z)$, $\text{Im}(z)$ and $|z|$.

Solution

Here $\text{Re}(z) = 3$, $\text{Im}(z) = -2$ and $|z| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$.

Complex conjugate

If $z = a + ib$ is any complex number then the complex conjugate of z is denoted by z^* and is defined by $z^* = a - ib$. (Sometimes the notation $\bar{z}$ is used instead of z^* to denote the conjugate). For example if $z = 2 - 3\text{i}$ then $z^* = 2 + 3\text{i}$. If z is entirely real then $z^* = z$ whereas if z is entirely imaginary then $z^* = -z$. E.g. if $z = 17\text{i}$ then $z^* = -17\text{i}$. In fact the following relationships are easily obtained:

$$\text{Re}(z) = \frac{z + z^*}{2} \qquad \text{and} \qquad \text{Im}(z) = \frac{\text{i}(z^* - z)}{2}$$

Task If $z = -2 + \text{i}$ find expressions for $\text{Re}(z^*)$ and $\text{Im}(\text{i}(z^* - z))$.

Hint: first find z^*, $z^* - z$, and $\text{i}(z^* - z)$:

Your solution

Answer

$\text{Re}(z^*) = -2$ and $\text{Im}(\text{i}(z^* - z)) = 0$

2. The algebra of complex numbers

Complex numbers are added, subtracted, multiplied and divided in much the same way as these operations are carried out for real numbers.

Addition and subtraction of complex numbers

Let z and w be any two complex numbers

$$z = a + \text{i}b \qquad w = c + \text{i}d$$

then

$$z + w = (a + c) + \text{i}(b + d) \qquad z - w = (a - c) + \text{i}(b - d)$$

For example if $z = 2 - 3\text{i}$, $w = -4 + 2\text{i}$ then

$$z + w = \{2 + (-4)\} + \{(-3) + 2\}\text{i} = -2 - \text{i} \qquad z - w = \{2 - (-4)\} + \{(-3) - 2\}\text{i} = 6 - 5\text{i}$$

Multiplying one complex number by another

In multiplication we proceed using an obvious approach: again consider any two complex numbers $z = a + \text{i}b$ and $w = c + \text{i}d$. Then

$$\begin{aligned} zw &= (a + \text{i}b)(c + \text{i}d) \\ &= ac + a\text{i}d + \text{i}bc + \text{i}^2bd \end{aligned}$$

obtained in the usual way by multiplying all the terms in one bracket by all the terms in the other bracket. Now we use the fundamental relation $\text{i}^2 = -1$ so that

$$\begin{aligned} zw &= ac + a\text{i}d + \text{i}bc - bd \\ &= ac - bd + \text{i}(ad + bc) \end{aligned}$$

where we have re-grouped terms with the 'i' symbol and terms without the 'i' symbol separately. These are the real and imaginary parts of the product zw respectively. A numerical example will confirm the approach. If $z = 2 - 3\text{i}$ and $w = -4 + 2\text{i}$ then

$$\begin{aligned} zw &= (2-3\text{i})(-4+2\text{i}) \\ &= 2(-4) + 2(2\text{i}) - 3\text{i}(-4) - 3\text{i}(2\text{i}) \\ &= -8 + 4\text{i} + 12\text{i} - 6\text{i}^2 \\ &= -8 + 16\text{i} + 6 \\ &= -2 + 16\text{i} \end{aligned}$$

Task

If $z = -2 + \text{i}$ and $w = 3 + 2\text{i}$ find expressions for

(a) $z + 2w$, (b) $|z - w|$ and (c) zw

Your solution

(a)

Answer

$z + 2w = 4 + 5\text{i}$

Your solution

(b) Hint: you should find that $z - w = -5 - \text{i}$

Answer

$|z - w| = \sqrt{(-5)^2 + (-1)^2} = \sqrt{26}$

Your solution

(c)

Answer

$zw = -6 + 3\text{i} - 4\text{i} + 2\text{i}^2 = -8 - \text{i}$

In general the square of a complex number is not necessarily a positive real number; it may not even be real at all. For example if $z = -2 + \text{i}$ then

$$z^2 = (-2+\text{i})^2 = 4 - 4\text{i} + \text{i}^2 = 4 - 4\text{i} - 1 = 3 - 4\text{i}$$

However, the product of a complex number with its conjugate is always a non-negative real number. If $z = a + \text{i}b$ then

$$\begin{aligned} zz^* &= (a + \text{i}b)(a - \text{i}b) \\ &= a^2 - a(\text{i}b) + (\text{i}b)a - \text{i}^2b^2 \\ &= a^2 - \text{i}^2b^2 \\ &= a^2 + b^2 \qquad \text{since } \text{i}^2 = -1 \end{aligned}$$

For example, if $z = 2 + \text{i}$ then

$$zz^* = (2 + \text{i})(2 - \text{i}) = 4 + 1 = 5$$

Task

Show, for any complex number $z = a + \text{i}b$ that $zz^* = |z|^2$.

Your solution

Answer

By definition $|z| = \sqrt{a^2 + b^2}$, so that $|z|^2 = a^2 + b^2$. Now $zz^* = a^2 + b^2$ so that $zz^* = |z|^2$.

Dividing one complex number by another

Here we consider the operation of dividing one complex number $z = a + \text{i}b$ by another, $w = c + \text{i}d$:

$$\frac{z}{w} = \frac{a + \text{i}b}{c + \text{i}d}$$

We wish to simplify the right-hand side into the standard form of a complex number (this is called the **Cartesian form)**:

(Real part) + i (Imaginary part)

or the equivalent:

(Real part) + (Imaginary part) i

To do this we multiply 'top and bottom' by the complex conjugate of the bottom (the denominator), that is, by $c - \text{i}d$ (this is called **rationalising**):

$$\frac{z}{w} = \frac{a + \text{i}b}{c + \text{i}d} = \frac{a + \text{i}b}{c + \text{i}d} \times \frac{c - \text{i}d}{c - \text{i}d}$$

and then carry out the multiplication, top and bottom:

$$\begin{aligned}\frac{z}{w} &= \frac{(ac + bd) + \text{i}(bc - ad)}{c^2 + d^2} \\ &= \left(\frac{ac + bd}{c^2 + d^2}\right) + \text{i}\left(\frac{bc - ad}{c^2 + d^2}\right)\end{aligned}$$

which is now in the required form.

The reason for rationalising is to get a real number in the denominator since a complex number divided by a real number is easy to evaluate.

Example 2

Find $\dfrac{z}{w}$ if $z = 2 - 3\text{i}$ and $w = 2 + \text{i}$.

Solution

$$\begin{aligned}
\frac{z}{w} = \frac{2-3\text{i}}{2+\text{i}} &= \frac{(2-3\text{i})\times(2-\text{i})}{(2+\text{i})\times(2-\text{i})} && \text{rationalising} \\
&= \frac{4-3+\text{i}(-6-2)}{4+1} && \text{multiplying out} \\
&= \frac{1}{5} - \frac{8}{5}\text{i} && \text{dividing through}
\end{aligned}$$

Task

If $z = 3 - \text{i}$ and $w = 1 + 3\text{i}$ find $\dfrac{2z+3w}{2z-3w}$.

Your solution

Answer

$$\begin{aligned}
\frac{2z+3w}{2z-3w} = \frac{9+7\text{i}}{3-11\text{i}} &= \frac{(9+7\text{i})(3+11\text{i})}{(3-11\text{i})(3+11\text{i})} \\
&= \frac{27-77+(21+99)\text{i}}{9+121} \\
&= -\frac{50}{130} + \frac{120}{130}\text{i} = -\frac{5}{13} + \frac{12}{13}\text{i}
\end{aligned}$$

Exercises

1. If $z = 2 - \text{i}$, $w = 3 + 4\text{i}$ find expressions (in standard Cartesian form) for
   (a) $z - 3w$, (b) zw^* (c) $\left(\dfrac{z}{w}\right)^*$ (d) $\left|\dfrac{z}{w}\right|$
2. Verify the following statements for general complex numbers $z = a + \text{i}b$ and $w = c + \text{i}d$
   (a) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ (b) $(zw)^* = z^*w^*$ (c) $\text{Re}(z) = \dfrac{z + z^*}{2}$ (d) $\text{Im}(z) = \dfrac{\text{i}(z^* - z)}{2}$.
3. Find z such that $zz^* + 3(z - z^*) = 13 + 12\text{i}$

Answers

1. (a) $-7 - 13\text{i}$ (b) $2 - 11\text{i}$ (c) $\dfrac{2}{25} + \dfrac{11}{25}\text{i}$ (d) $\dfrac{\sqrt{5}}{5}$
2. Note that since $z^* - z$ is **imaginary** then $\text{i}(z^* - z)$ is **real**!
3. $z = \pm 3 + 2\text{i}$

3. Solutions of polynomial equations

With the introduction of complex numbers we can now obtain solutions to those polynomial equations which may have real solutions, complex solutions or a combination of real and complex solutions. For example, the simple quadratic equation:

$$x^2 + 16 = 0 \qquad \text{can be rearranged:} \qquad x^2 = -16$$

and then taking square roots:

$$x = \pm\sqrt{-16} = \pm 4\sqrt{-1} = \pm 4\text{i}$$

where we are replacing $\sqrt{-1}$ by the symbol 'i'.
This approach can be extended to the general quadratic equation

$$ax^2 + bx + c = 0 \qquad \text{with roots} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that for example, if

$$3x^2 + 2x + 2 = 0$$

then solving for x:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(3)(2)}}{2(3)} \\ &= \frac{-2 \pm \sqrt{-20}}{6} = \frac{-2 \pm \text{i}\sqrt{20}}{6} \end{aligned}$$

so, (as $\dfrac{\sqrt{20}}{6} = \dfrac{2\sqrt{5}}{6} = \dfrac{\sqrt{5}}{3}$), the two roots are $-\dfrac{1}{3} + \dfrac{\sqrt{5}}{3}\text{i}$ and $-\dfrac{1}{3} - \dfrac{\sqrt{5}}{3}\text{i}$.
In this example we see that the two solutions (roots) are **complex conjugates** of each other. In fact this will always be the case if the polynomial equation has **real** coefficients: that is, if any complex roots occur they will always occur in complex conjugate pairs.

Key Point 5

Complex roots to polynomial equations having **real** coefficients

always occur in **complex conjugate pairs**

Example 3

Given that $x = 3 - 2\mathrm{i}$ is one root of the cubic equation $x^3 - 7x^2 + 19x - 13 = 0$ find the other two roots.

Solution

Since the coefficients of the equation are real and $3 - 2\mathrm{i}$ is a root then its complex conjugate $3 + 2\mathrm{i}$ is also a root which implies that $x - (3 - 2\mathrm{i})$ and $x - (3 + 2\mathrm{i})$ are factors of the given cubic expression. Multiplying together these two factors:

$$(x - (3 - 2\mathrm{i}))(x - (3 + 2\mathrm{i})) = x^2 - x(3 - 2\mathrm{i}) - x(3 + 2\mathrm{i}) + 13 = x^2 - 6x + 13$$

So $x^2 - 6x + 13$ is a quadratic factor of the cubic equation. The remaining factor **must** take the form $(x + a)$ where a is real, since only one more linear factor of the cubic equation is required, and so we write:

$$x^3 - 7x^2 + 19x - 13 = (x^2 - 6x + 13)(x + a)$$

By inspection (consider for example the constant terms), it is clear that $a = -1$ so that the final factor is $(x - 1)$, implying that the original cubic equation has a root at $x = 1$.

Exercises

1. Find the roots of the equation $x^2 + 2x + 2 = 0$.
2. If i is one root of the cubic equation $x^3 + 2x^2 + x + 2 = 0$ find the two other roots.
3. Find the complex number z if $2z + z^* + 3\mathrm{i} + 2 = 0$.
4. If $z = \cos\theta + \mathrm{i}\sin\theta$ show that $\dfrac{z}{z^*} = \cos 2\theta + \mathrm{i}\sin 2\theta$.

Answers 1. $x = -1 \pm \mathrm{i}$ 2. $-\mathrm{i},\ -2$ 3. $-\dfrac{2}{3} - 3\mathrm{i}$