The Vector Product

9.4

Introduction

In this Section we describe how to find the **vector product** of two vectors. Like the scalar product, its definition may seem strange when first met but the definition is chosen because of its many applications. When vectors are multiplied using the vector product the result is always a vector.

Prerequisites

Before starting this Section you should ...

- know that a vector can be represented as a directed line segment
- know how to express a vector in Cartesian form
- know how to evaluate 3×3 determinants

Learning Outcomes

On completion you should be able to ...

- use the right-handed screw rule
- calculate the vector product of two given vectors
- use determinants to calculate the vector product of two vectors given in Cartesian form

1. The right-handed screw rule

To understand how the vector product is formed it is helpful to consider first the right-handed screw rule. Consider the two vectors $\underline{a}$ and $\underline{b}$ shown in Figure 37.

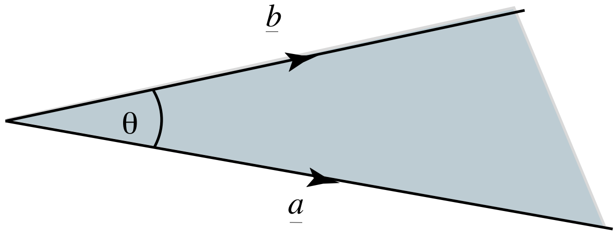


Figure 37

The two vectors lie in a plane; this plane is shaded in Figure 37. Figure 38 shows the same two vectors and the plane in which they lie together with a unit vector, denoted $\hat{\underline{e}}$, which is perpendicular to this plane. Imagine turning a right-handed screw, aligned along $\hat{\underline{e}}$, in the direction from $\underline{a}$ towards $\underline{b}$ as shown. A right-handed screw is one which when turned clockwise enters the material into which it is being screwed (most screws are of this kind). You will see from Figure 38 that the screw will advance in the direction of $\hat{\underline{e}}$.

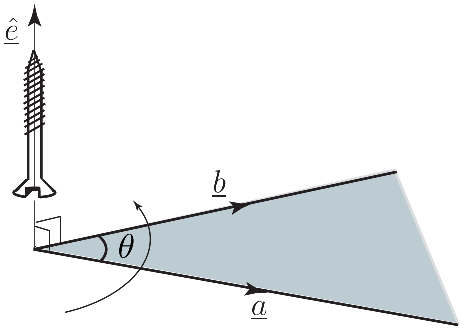


Figure 38

On the other hand, if the right-handed screw is turned from $\underline{b}$ towards $\underline{a}$ the screw will retract in the direction of $\hat{\underline{f}}$ as shown in Figure 39.

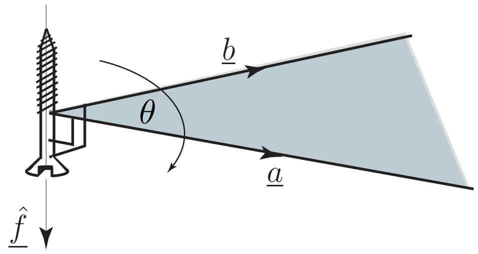


Figure 39

We are now in a position to describe the vector product.

2. Definition of the vector product

We define the vector product of $\underline{a}$ and $\underline{b}$, written $\underline{a}\times\underline{b}$, as

$$\underline{a}\times\underline{b} = |\underline{a}|\,|\underline{b}|\,\sin\theta\,\hat{\underline{e}}$$

By inspection of this formula note that this is a **vector** of magnitude $|\underline{a}|\,|\underline{b}|\,\sin\theta$ in the direction of the vector $\hat{\underline{e}}$, where $\hat{\underline{e}}$ is a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$ in the sense defined by the right-handed screw rule. The quantity $\underline{a}\times\underline{b}$ is read as "$\underline{a}$ cross $\underline{b}$" and is sometimes referred to as the **cross product**. The angle is chosen to lie between 0 and π. See Figure 40.

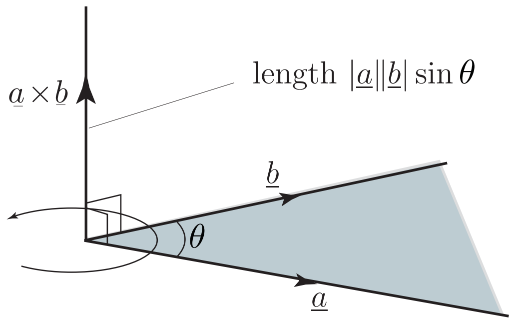


Figure 40: $\underline{a}\times\underline{b}$ is perpendicular to the plane containing $\underline{a}$ and $\underline{b}$

Formally we have

Key Point 17

The vector product of $\underline{a}$ and $\underline{b}$ is: $\underline{a}\times\underline{b} = |\underline{a}|\,|\underline{b}|\,\sin\theta\,\hat{\underline{e}}$
The modulus of the vector product is: $|\underline{a}\times\underline{b}| = |\underline{a}|\,|\underline{b}|\,\sin\theta$

Note that $|\underline{a}|\,|\underline{b}|\,\sin\theta$ gives the modulus of the vector product whereas $\hat{\underline{e}}$ gives its direction.
Now study Figure 41 which is used to illustrate the calculation of $\underline{b}\times\underline{a}$. In particular note the direction of $\underline{b}\times\underline{a}$ arising through the application of the right-handed screw rule.
We see that $\underline{a}\times\underline{b}$ is not equal to $\underline{b}\times\underline{a}$ because their directions are **opposite**. In fact $\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$.

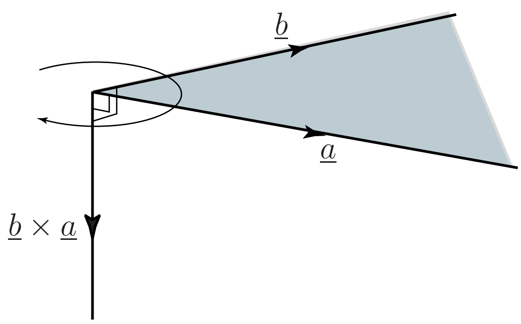


Figure 41: $\underline{b}\times\underline{a}$ is in the opposite direction to $\underline{a}\times\underline{b}$

Example 15

If $\underline{a}$ and $\underline{b}$ are parallel, show that $\underline{a} \times \underline{b} = \underline{0}$.

Solution

If $\underline{a}$ and $\underline{b}$ are parallel then the angle θ between them is zero. Consequently $\sin\theta = 0$ from which it follows that $\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}|\,\sin\theta\,\hat{\underline{e}} = \underline{0}$. Note that the result, $\underline{0}$, is the **zero vector**.

Note in particular the important results which follow:

Key Point 18

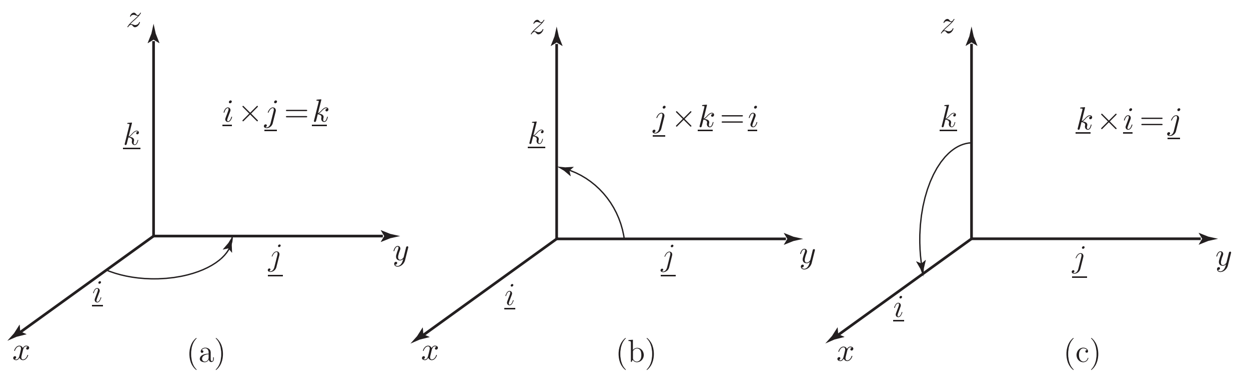
$$\underline{i} \times \underline{i} = \underline{0} \qquad \underline{j} \times \underline{j} = \underline{0} \qquad \underline{k} \times \underline{k} = \underline{0}$$

Example 16

Show that $\underline{i} \times \underline{j} = \underline{k}$ and find expressions for $\underline{j} \times \underline{k}$ and $\underline{k} \times \underline{i}$.

Solution

Note that $\underline{i}$ and $\underline{j}$ are perpendicular so that the angle between them is 90°. Also, the vector $\underline{k}$ is perpendicular to both $\underline{i}$ and $\underline{j}$. Using Key Point 17, the modulus of $\underline{i} \times \underline{j}$ is $(1)(1)\sin 90^\circ = 1$. So $\underline{i}\times\underline{j}$ is a unit vector.The unit vector perpendicular to $\underline{i}$ and $\underline{j}$ in the sense defined by the right-handed screw rule is $\underline{k}$ as shown in Figure 42(a). Therefore $\underline{i} \times \underline{j} = \underline{k}$ as required.



Similarly you should verify that $\underline{j} \times \underline{k} = \underline{i}$ (Figure 42(b)) and $\underline{k} \times \underline{i} = \underline{j}$ (Figure 42(c)).

Key Point 19

$\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}, \qquad \underset{\sim}{j} \times \underset{\sim}{k} = \underset{\sim}{i}, \qquad \underset{\sim}{k} \times \underset{\sim}{i} = \underset{\sim}{j} \qquad \underset{\sim}{j} \times \underset{\sim}{i} = -\underset{\sim}{k}, \qquad \underset{\sim}{k} \times \underset{\sim}{j} = -\underset{\sim}{i}, \qquad \underset{\sim}{i} \times \underset{\sim}{k} = -\underset{\sim}{j}$

To help remember these results you might like to think of the vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ written in alphabetical order like this:

$$\underset{\sim}{i} \quad \underset{\sim}{j} \quad \underset{\sim}{k} \quad \underset{\sim}{i} \quad \underset{\sim}{j} \quad \underset{\sim}{k}$$

Moving left to right yields a positive result: e.g. $\underset{\sim}{k} \times \underset{\sim}{i} = \underset{\sim}{j}$.

Moving right to left yields a negative result: e.g. $\underset{\sim}{j} \times \underset{\sim}{i} = -\underset{\sim}{k}$.

3. A formula for finding the vector product

We can use the results in Key Point 19 to develop a formula for finding the vector product of two vectors given in Cartesian form: Suppose $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$ then

$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{b} &= (a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}) \times (b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}) \\
&= a_1\underset{\sim}{i} \times (b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}) \\
&\qquad + \quad a_2\underset{\sim}{j} \times (b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}) \\
&\qquad\qquad + \quad a_3\underset{\sim}{k} \times (b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}) \\
&= a_1b_1(\underset{\sim}{i} \times \underset{\sim}{i}) + a_1b_2(\underset{\sim}{i} \times \underset{\sim}{j}) + a_1b_3(\underset{\sim}{i} \times \underset{\sim}{k}) \\
&\qquad + \quad a_2b_1(\underset{\sim}{j} \times \underset{\sim}{i}) + a_2b_2(\underset{\sim}{j} \times \underset{\sim}{j}) + a_2b_3(\underset{\sim}{j} \times \underset{\sim}{k}) \\
&\qquad\qquad + \quad a_3b_1(\underset{\sim}{k} \times \underset{\sim}{i}) + a_3b_2(\underset{\sim}{k} \times \underset{\sim}{j}) + a_3b_3(\underset{\sim}{k} \times \underset{\sim}{k})
\end{aligned}$$

Using Key Point 19, this expression simplifies to

$$\underset{\sim}{a} \times \underset{\sim}{b} = (a_2b_3 - a_3b_2)\underset{\sim}{i} - (a_1b_3 - a_3b_1)\underset{\sim}{j} + (a_1b_2 - a_2b_1)\underset{\sim}{k}$$

This gives us Key Point 20:

Key Point 20

If $\quad \underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k} \quad$ and $\quad \underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k} \quad$ then

$$\underset{\sim}{a} \times \underset{\sim}{b} = (a_2b_3 - a_3b_2)\underset{\sim}{i} - (a_1b_3 - a_3b_1)\underset{\sim}{j} + (a_1b_2 - a_2b_1)\underset{\sim}{k}$$

Example 17

Evaluate the vector product $\underline{a} \times \underline{b}$ if $\underline{a} = 3\underline{i} - 2\underline{j} + 5\underline{k}$ and $\underline{b} = 7\underline{i} + 4\underline{j} - 8\underline{k}$.

Solution

Identifying $a_1 = 3$, $a_2 = -2$, $a_3 = 5$, $b_1 = 7$, $b_2 = 4$, $b_3 = -8$ we find

$$\begin{aligned}\underline{a} \times \underline{b} &= ((-2)(-8) - (5)(4))\underline{i} - ((3)(-8) - (5)(7))\underline{j} + ((3)(4) - (-2)(7))\underline{k} \\ &= -4\underline{i} + 59\underline{j} + 26\underline{k}\end{aligned}$$

Task

Use Key Point 20 to find the vector product of $\underline{p} = 3\underline{i} + 5\underline{j}$ and $\underline{q} = 2\underline{i} - \underline{j}$.

Note that in this case there are no $\underline{k}$ components so a_3 and b_3 are both zero:

Your solution

$\underline{p} \times \underline{q} =$

Answer

$-13\underline{k}$

4. Using determinants to evaluate a vector product

Evaluation of a vector product using the formula in Key Point 20 is very cumbersome. A more convenient and easily remembered method is to use determinants. Recall from Workbook 7 that, for a 3×3 determinant,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The vector product of two vectors $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ can be found by evaluating the determinant:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

in which $\underline{i}$, $\underline{j}$ and $\underline{k}$ are (temporarily) treated as if they were scalars.

Key Point 21

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}(a_2b_3 - a_3b_2) - \underline{j}(a_1b_3 - a_3b_1) + \underline{k}(a_1b_2 - a_2b_1)$$

Example 18

Find the vector product of $\underline{a} = 3\underline{i} - 4\underline{j} + 2\underline{k}$ and $\underline{b} = 9\underline{i} - 6\underline{j} + 2\underline{k}$.

Solution

We have $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & 2 \\ 9 & -6 & 2 \end{vmatrix}$ which, when evaluated, gives

$$\underline{a} \times \underline{b} = \underline{i}(-8 - (-12)) - \underline{j}(6 - 18) + \underline{k}(-18 - (-36)) = 4\underline{i} + 12\underline{j} + 18\underline{k}$$

Example 19

The area A_T of the triangle shown in Figure 43 is given by the formula $A_T = \frac{1}{2}bc\sin\alpha$. Show that an equivalent formula is $A_T = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

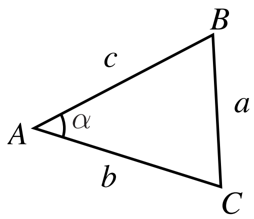


Figure 43

Solution

We use the definition of the vector product $|\overrightarrow{AB} \times \overrightarrow{AC}| = |\overrightarrow{AB}|\,|\overrightarrow{AC}|\sin\alpha$.

Since α is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and $|\overrightarrow{AB}| = c$ and $|\overrightarrow{AC}| = b$, the required result follows immediately:

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}c \cdot b \cdot \sin\alpha.$$

Moments

The **moment** (or **torque**) of the force $\underline{F}$ about a point O is defined as

$$\underline{M}_o = \underline{r} \times \underline{F}$$

where $\underline{r}$ is a position vector from O to *any* point on the line of action of $\underline{F}$ as shown in Figure 44.

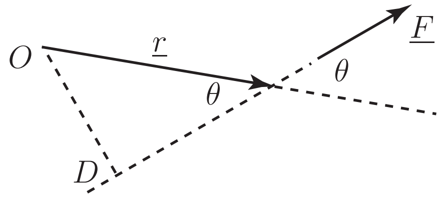


Figure 44

It may seem strange that any point on the line of action may be taken but it is easy to show that exactly the same vector $\underline{M}_o$ is always obtained.

By the properties of the cross product the direction of $\underline{M}_o$ is perpendicular to the plane containing $\underline{r}$ and $\underline{F}$ (i.e. out of the paper). The magnitude of the moment is

$$|\underline{M}_0| = |\underline{r}||\underline{F}|\sin\theta.$$

From Figure 32, $|\underline{r}|\sin\theta = D$. Hence $|\underline{M}_o| = D|\underline{F}|$. This would be the same no matter which point on the line of action of $\underline{F}$ was chosen.

Example 20

Find the moment of the force given by $\underline{F} = 3\underline{i} + 4\underline{j} + 5\underline{k}$ (N) acting at the point $(14, -3, 6)$ about the point $P(2, -2, 1)$.

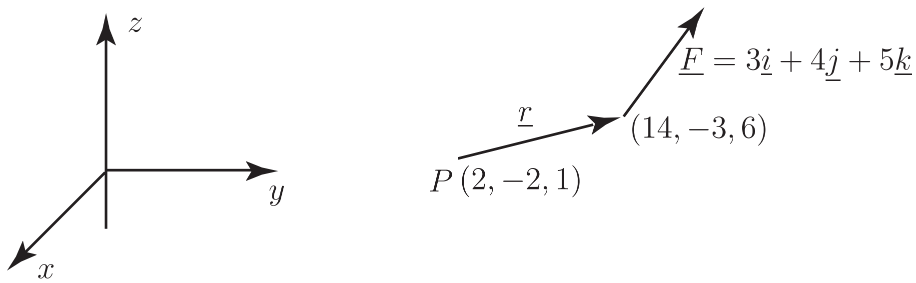


Figure 45

Solution

The vector $\underline{r}$ can be any vector from the point P to any point on the line of action of $\underline{F}$. Choosing $\underline{r}$ to be the vector connecting P to $(14, -3, 6)$ (and measuring distances in metres) we have:

$$\underline{r} = (14-2)\underline{i} + (-3-(-2))\underline{j} + (6-1)\underline{k} = 12\underline{i} - \underline{j} + 5\underline{k}.$$

The moment is $\quad \underline{M} = \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 12 & -1 & 5 \\ 3 & 4 & 5 \end{vmatrix} = -25\underline{i} - 45\underline{j} + 51\underline{k}$ (N m)

Exercises

1. Show that if $\underline{a}$ and $\underline{b}$ are parallel vectors then their vector product is the zero vector.

2. Find the vector product of $\underline{p} = -2\underline{i} - 3\underline{j}$ and $\underline{q} = 4\underline{i} + 7\underline{j}$.

3. If $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{b} = 4\underline{i} + 3\underline{j} + 2\underline{k}$ find $\underline{a} \times \underline{b}$. Show that $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$.

4. Points A, B and C have coordinates $(9, 1, -2)$, (3,1,3), and $(1, 0, -1)$ respectively. Find the vector product $\overrightarrow{AB} \times \overrightarrow{AC}$.

5. Find a vector which is perpendicular to both of the vectors $\underline{a} = \underline{i} + 2\underline{j} + 7\underline{k}$ and $\underline{b} = \underline{i} + \underline{j} - 2\underline{k}$. Hence find a unit vector which is perpendicular to both $\underline{a}$ and $\underline{b}$.

6. Find a vector which is perpendicular to the plane containing $6\underline{i} + \underline{k}$ and $2\underline{i} + \underline{j}$.

7. For the vectors $\underline{a} = 4\underline{i} + 2\underline{j} + \underline{k}$, $\underline{b} = \underline{i} - 2\underline{j} + \underline{k}$, and $\underline{c} = 3\underline{i} - 3\underline{j} + 4\underline{k}$, evaluate both $\underline{a} \times (\underline{b} \times \underline{c})$ and $(\underline{a} \times \underline{b}) \times \underline{c}$. Deduce that, in general, the vector product is not associative.

8. Find the area of the triangle with vertices at the points with coordinates $(1, 2, 3)$, $(4, -3, 2)$ and $(8, 1, 1)$.

9. For the vectors $\underline{r} = \underline{i} + 2\underline{j} + 3\underline{k}$, $\underline{s} = 2\underline{i} - 2\underline{j} - 5\underline{k}$, and $\underline{t} = \underline{i} - 3\underline{j} - \underline{k}$, evaluate
   (a) $(\underline{r} \cdot \underline{t})\underline{s} - (\underline{s} \cdot \underline{t})\underline{r}$. (b) $(\underline{r} \times \underline{s}) \times \underline{t}$. Deduce that $(\underline{r} \cdot \underline{t})\underline{s} - (\underline{s} \cdot \underline{t})\underline{r} = (\underline{r} \times \underline{s}) \times \underline{t}$.

Answers

1. This uses the fact that $\sin 0 = 0$.
2. $-2\underline{k}$
3. $-5\underline{i} + 10\underline{j} - 5\underline{k}$
4. $5\underline{i} - 34\underline{j} + 6\underline{k}$
5. $-11\underline{i} + 9\underline{j} - \underline{k}$, $\quad \dfrac{1}{\sqrt{203}}(-11\underline{i} + 9\underline{j} - \underline{k})$
6. $-\underline{i} + 2\underline{j} + 6\underline{k}$ for example.
7. $7\underline{i} - 17\underline{j} + 6\underline{k}$, $\quad -42\underline{i} - 46\underline{j} - 3\underline{k}$. These are different so the vector product is **not** associative.
8. $\frac{1}{2}\sqrt{1106}$
9. Each gives $-29\underline{i} - 10\underline{j} + \underline{k}$