Contents 9

Vectors

Learning outcomes

In this Workbook you will learn what a vector is and how to combine vectors together using the triangle law. You will be able to represent a vector by its Cartesian components. You will be able to multiply vectors together using either the scalar product or the vector product. You will be able to apply your knowledge of vectors to solve problems involving forces and to geometric problems involving lines and planes.

Basic Concepts of Vectors

9.1

Introduction

In engineering, frequent reference is made to physical quantities, such as force, speed and time. For example, we talk of the speed of a car, and the force in a compressed spring. It is useful to separate these physical quantities into two types. Quantities of the first type are known as **scalars**. These can be fully described by a single number known as the **magnitude**. Quantities of the second type are those which require the specification of a **direction**, in addition to a magnitude, before they are completely described. These are known as **vectors**. Special methods have been developed for handling vectors in calculations, giving rise to subjects such as vector algebra, vector geometry and vector calculus. Quantities that are vectors must be manipulated according to certain rules, which are described in this and subsequent Sections.

Prerequisites

Before starting this Section you should . . .

- be familiar with all the basic rules of algebra

Learning Outcomes

On completion you should be able to . . .

- categorize a number of common physical quantities as scalar or vector

- represent vectors by directed line segments

- combine, or add, vectors using the triangle law

- resolve a vector into two perpendicular components

1. Introduction

It is useful to separate physical quantities into two types: the first are called **scalars**; the second are known as **vectors**. A scalar is a quantity that can be described by a single number which can be positive, negative or zero. An example of a scalar quantity is the mass of an object, so we might state that 'the mass of the stone is 3 kg'. It is important to give the units in which the quantity is measured. Obvious examples of scalars are temperature and length, but there are many other engineering applications in which scalars play an important role. For example, speed, work, voltage and energy are all scalars.

On the other hand, vectors are quantities which require the specification of a **magnitude** and a **direction**. An example of a vector quantity is the force applied to an object to make it move. When the object shown in Figure 1 is moved by applying a force to it we achieve different effects depending on the direction of the force.


Figure 1: A force is a vector quantity

In order to specify the force completely we must state not only its magnitude (its 'strength') but also the direction in which the force acts. For example we might state that 'a force of 5 newtons is applied vertically from above'. Clearly this force would have a different effect from one applied horizontally. The *direction* in which the force acts is crucial.

There are many engineering applications where vectors are important. Force, acceleration, velocity, electric and magnetic fields are all described by vectors. Furthermore, when computer software is written to control the position of a robot, the position is described by vectors.

Sometimes confusion can arise because words used in general conversation have specific technical meanings when used in engineering calculations. An example is the use of the words 'speed' and 'velocity'. In everyday conversation these words have the same meaning and are used interchangeably. However in more precise language they are not the same. **Speed** is a scalar quantity described by giving a single number in appropriate units; for example 'the speed of the car is 40 kilometres per hour'. On the other hand **velocity** is a vector quantity and must be specified by giving a direction as well. For example 'the velocity of the aircraft is 20 metres per second due north'.

In engineering calculations, the words speed and velocity cannot be used interchangeably. Similar problems arise from use of the words 'mass' and 'weight'. **Mass** is a scalar which describes the amount of substance in an object. The unit of mass is the kilogramme. **Weight** is a vector, the direction of which is vertically downwards because weight arises from the action of gravity. The unit of weight is the newton. **Displacement** and **distance** are related quantities which can also cause confusion. Whereas distance is a scalar, displacement is 'directed distance', that is, distance together with a specified direction. So, referring to Figure 2, if an object is moved from point A to point B, we can state that the distance moved is 10 metres, but the displacement is 10 metres *in the direction from A to B*.

You will meet many other quantities in the course of your studies and it will be helpful to know which are vectors and which are scalars. Some common quantities and their type are listed in Table 1. The S.I. units in which these are measured are also shown.
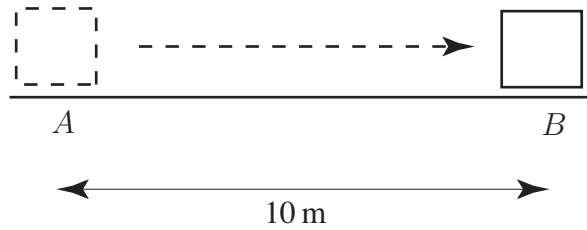
Figure 2: Displacement means directed distance

Table 1. Some common scalar and vector quantities

quantity	type	S.I. unit
distance	scalar	metre, m
mass	scalar	kilogramme, kg
temperature	scalar	kelvin, K
pressure	scalar	pascal, Pa
work	scalar	joule, J
energy	scalar	joule, J
displacement	vector	metre m
force	vector	newton, N
velocity	vector	metres per second, m s^{-1}
acceleration	vector	metres per second per second, m s^{-2}

Exercise

State which of the following are scalars and which are vectors:
  (a) the volume of a petrol tank,
  (b) a length measured in metres,
  (c) a length measured in miles,
  (d) the angular velocity of a flywheel,
  (e) the relative velocity of two aircraft,
  (f) the work done by a force,
  (g) electrostatic potential,
  (h) the momentum of an atomic particle.

Answer

(a), (b), (c) (f), (g) are scalars.

(d), (e), and (h) are vectors

2. The mathematical description of vector quantities

Because a vector has a direction as well as a magnitude we can represent a vector by drawing a line. The length of the line represents the **magnitude** of the vector given some appropriate scale, and the direction of the line represents the **direction** of the vector. We call this representation a **directed line segment**. For example, Figure 3 shows a vector which represents a velocity of 3 m s^{-1} north-west. Note that the arrow on the vector indicates the direction required.
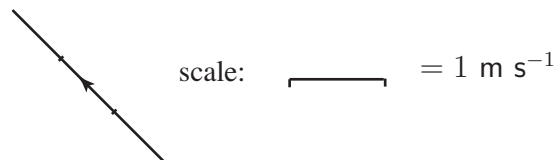
Figure 3: A vector quantity can be represented by drawing a line

More generally, Figure 4 shows an arbitrary vector quantity.
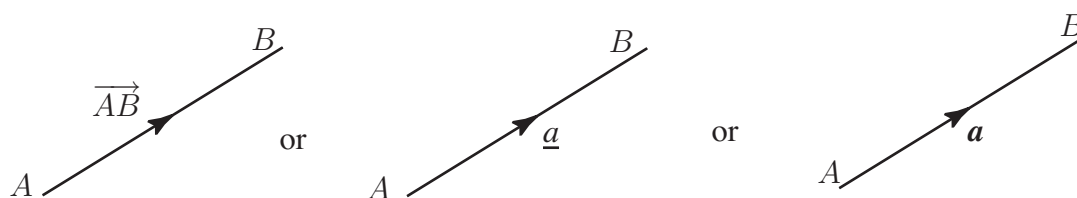
Figure 4: Representation of vectors

It is important when writing vectors to distinguish them from scalars. Various notations are used. In Figure 4 we emphasise that we are dealing with the vector from A to B by using an **arrow** and writing $\overrightarrow{AB}$. Often, in textbooks, vectors are indicated by using a **bold typeface** such as $\boldsymbol{a}$. It is difficult when handwriting to reproduce the bold face and so it is conventional to **underline** vector quantities and write $\underline{a}$ instead. So $\overrightarrow{AB}$ and $\underline{a}$ represent the same vector in Figure 4. We can also use the notation $\underline{AB}$. In general in this Workbook we will use underlining but we will also use the arrow notation where it is particularly helpful.

Example 1

Figure 5 shows an object pulled by a force of 10 N at an angle of $60°$ to the horizontal.
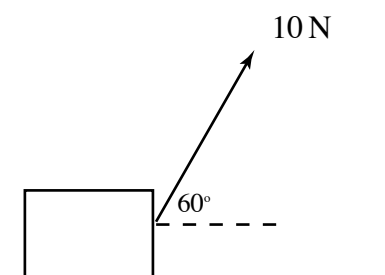
Figure 5
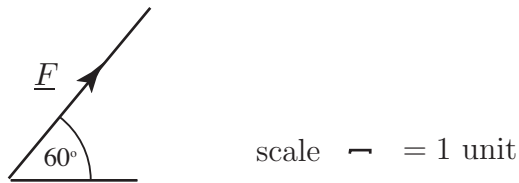
Show how this force can be represented by a vector.

When we wish to refer simply to the magnitude (or length) of a vector we write this using the **modulus** sign as $|\overrightarrow{AB}|$, or $|\underline{a}|$, or simply a (without the underline.)

In general two vectors are said to be **equal vectors** if they have the same magnitude and same direction. So, in Figure 7 the vectors $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are equal even though their locations differ.
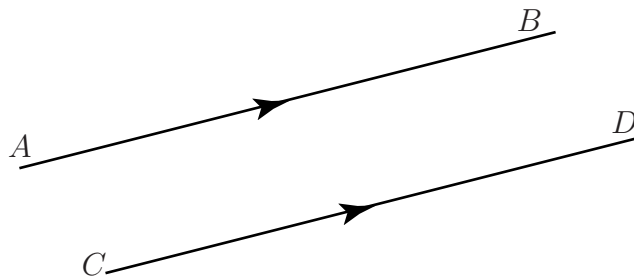
Figure 7: Equal vectors

This is a useful and important property of vectors: a vector is defined only by its direction and magnitude, not by its location in space. These vectors are often called **free** vectors.
The vector $-\underline{a}$ is a vector in the opposite direction to $\underline{a}$, but has the same magnitude as $\underline{a}$, as shown in Figure 8.
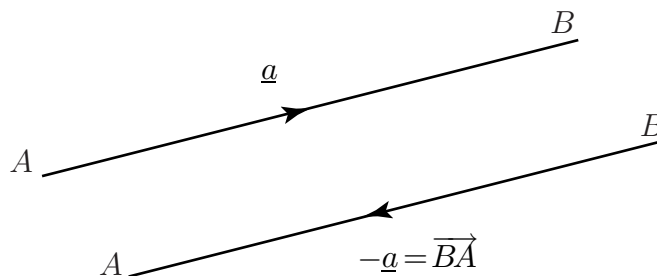

Figure 8

Geometrically, if $\underline{a} = \overrightarrow{AB}$ then $-\underline{a} = \overrightarrow{BA}$.

Exercises

1. An object is subject to two forces, one of 3 N vertically downwards, and one of 8 N, horizontally to the right. Draw a diagram representing these two forces as vectors.

2. Draw a diagram showing an arbitrary vector $\underline{F}$. On the diagram show the vector $-\underline{F}$.

3. Vectors $\underline{p}$ and $\underline{q}$ are equal vectors. Draw a diagram to represent $\underline{p}$ and $\underline{q}$.

4. If $\underline{F}$ is a vector, what is meant by F?

Answers
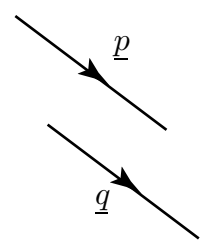
1.

8 N

3 N

2.

$\underline{F}$

$-\underline{F}$

3.

$\underline{p}$

$\underline{q}$
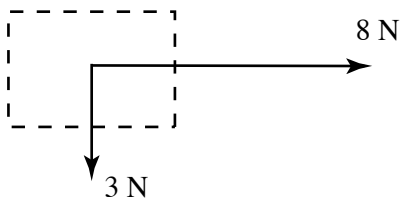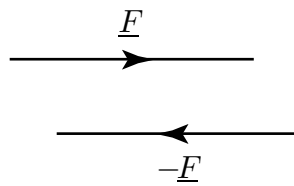
4. F is the magnitude of $\underline{F}$.

3. Addition of vectors

Vectors are added in a particular way known as the **triangle law**. To see why this law is appropriate to add them this way consider the following example:

Example: The route taken by an automated vehicle

An unmanned vehicle moves on tracks around a factory floor carrying components from the store at A to workers at C as shown in Figure 9.
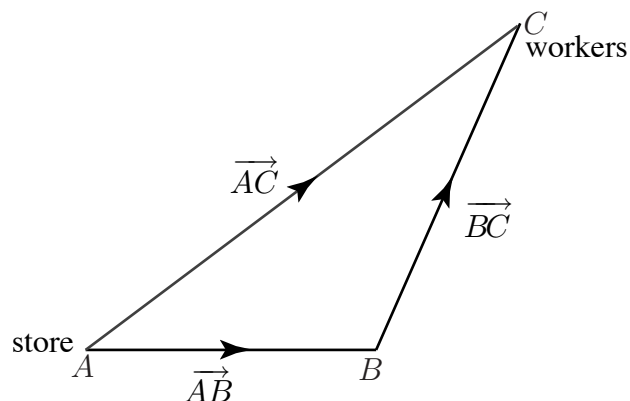
C
workers

$\overrightarrow{AC}$

$\overrightarrow{BC}$

store

A

$\overrightarrow{AB}$

B

Figure 9

The vehicle may arrive at C either directly or via an intermediate point B. The movement from A to B can be represented by a displacement vector $\overrightarrow{AB}$. Similarly, movement from B to C can be

represented by the displacement vector $\overrightarrow{BC}$, and movement from A to C can be represented by $\overrightarrow{AC}$. Since travelling from A to B and then B to C is equivalent to travelling directly from A to C we write

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

This is an example of the triangle law for adding vectors. We add vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ by placing the tail of $\overrightarrow{BC}$ at the head of $\overrightarrow{AB}$ and completing the third side of the triangle so formed $(\overrightarrow{AC})$.
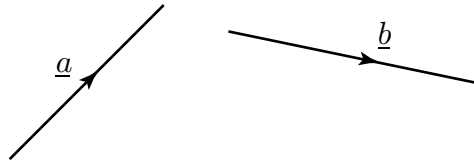
**Figure 10**: Two vectors $\underline{a}$ and $\underline{b}$

Consider the more general situation in Figure 10. Suppose we wish to add $\underline{b}$ to $\underline{a}$. To do this $\underline{b}$ is translated, keeping its direction and length unchanged, until its tail coincides with the head of $\underline{a}$. Then the sum $\underline{a} + \underline{b}$ is defined by the vector represented by the third side of the completed triangle, that is $\underline{c}$ in Figure 11. Note, from Figure 11, that we can write $\underline{c} = \underline{a} + \underline{b}$ since going along $\underline{a}$ and then along $\underline{b}$ is equivalent to going along $\underline{c}$.
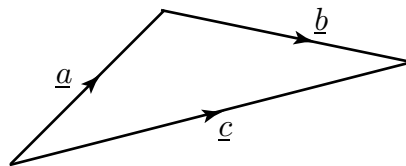
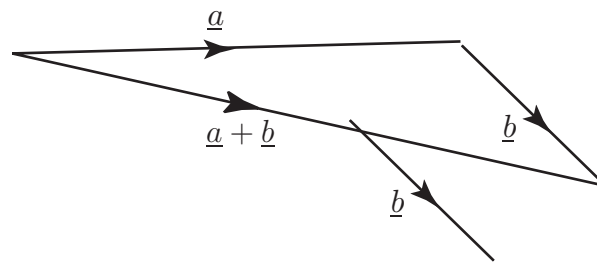**Figure 11**: Addition of the two vectors of Figure 10 using the triangle law

Task

Using vectors $\underline{a}$ and $\underline{b}$ shown below, draw a diagram to find $\underline{a} + \underline{b}$. Find also $\underline{b} + \underline{a}$. Is $\underline{a} + \underline{b}$ the same as $\underline{b} + \underline{a}$?
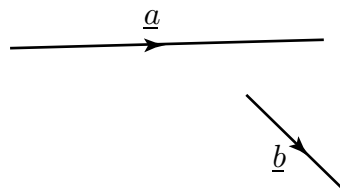
Your solution

Answer

yes $\underline{a} + \underline{b} = \underline{b} + \underline{a}$

It is possible, using the triangle law, to prove the following rules which apply to any three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$:

Key Point 1

Vector Addition

$\underline{a} + \underline{b} = \underline{b} + \underline{a}$ vector addition is commutative
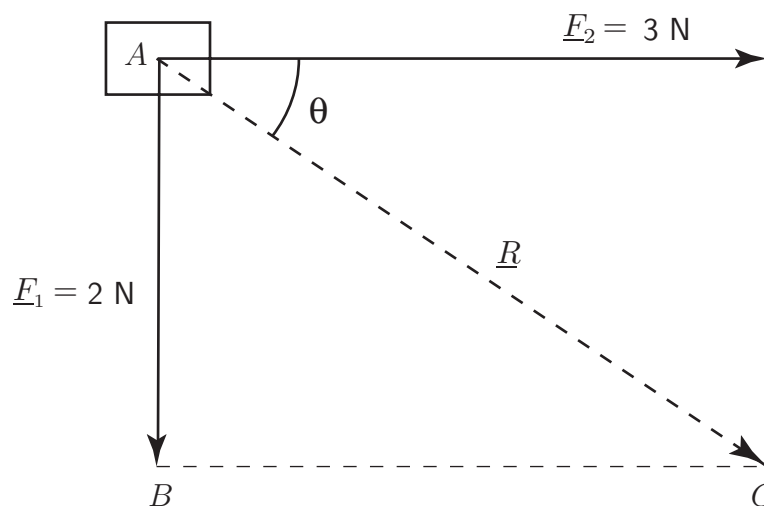
$\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$ vector addition is associative

Example: Resultant of two forces acting upon a body

A force $\underline{F}_1$ of 2 N acts vertically downwards, and a force $\underline{F}_2$ of 3 N acts horizontally to the right, upon the body shown in Figure 12.

Figure 12

We can use vector addition to find the combined effect or **resultant** of the two concurrent forces. (Concurrent means that the forces act through the same point.) Translating $\underline{F}_2$ until its tail touches

the head of $\underline{F}_1$, we complete the triangle ABC as shown. The vector represented by the third side is the resultant, $\underline{R}$. We write

$$\underline{R} = \underline{F}_1 + \underline{F}_2$$

and say that $\underline{R}$ is the **vector sum** of $\underline{F}_2$ and $\underline{F}_1$. The resultant force acts at an angle of θ below the horizontal where $\tan\theta = 2/3$, so that $\theta = 33.7°$, and has magnitude (given by Pythagoras' theorem) $\sqrt{13}\,$N.

Example: Resolving a force into two perpendicular directions

In the previous Example we saw that two forces acting upon a body can be replaced by a single force which has the same effect. It is sometimes useful to reverse this process and consider a single force as equivalent to two forces acting at right-angles to each other.

Consider the force $\underline{F}$ inclined at an angle θ to the horizontal as shown in Figure 13.
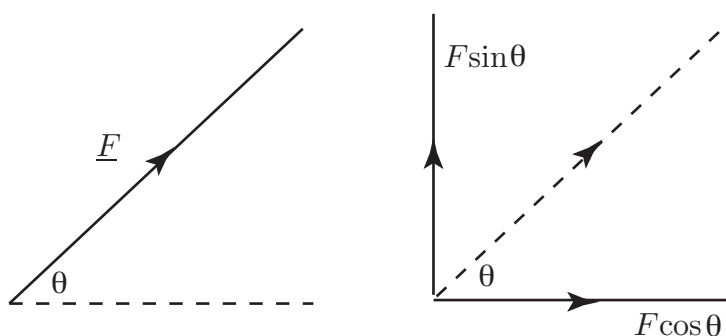

Figure 13

$\underline{F}$ can be replaced by two forces, one of magnitude $F\cos\theta$ and one of magnitude $F\sin\theta$ as shown. We say that $\underline{F}$ has been **resolved into two perpendicular components**. This is sensible because if we re-combine the two perpendicular forces of magnitudes $F\cos\theta$ and $F\sin\theta$ using the triangle law we find $\underline{F}$ to be the resultant force.

For example, Figure 14 shows a force of 5 N acting at an angle of $30°$ to the x axis. It can be resolved into two components, one directed along the x axis with magnitude $5\cos 30°$ and one perpendicular to this of magnitude $5\sin 30°$. Together, these two components have the same effect as the original force.

Figure 14
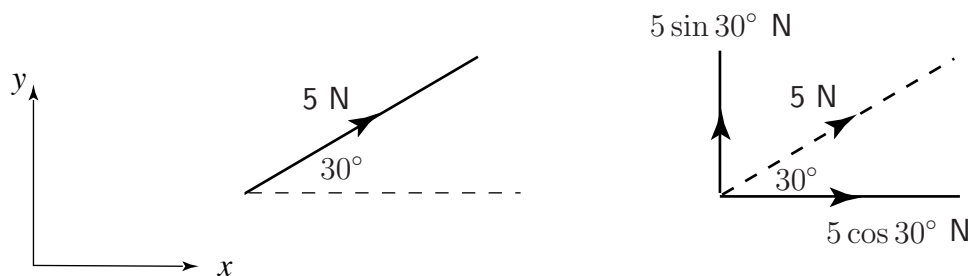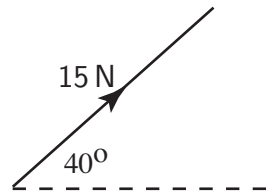
Task

Consider the force shown in the diagram below.

15 N
40°

Resolve this force into two perpendicular components, one horizontally to the right, and one vertically upwards.

Your solution

Answer

Horizontal component is $15\cos 40°$ N $= 11.50$ N; vertical component is $15\sin 40°$ N $= 9.64$ N

The need to **resolve** a vector along a given direction occurs in other areas. For example, as a police car or ambulance with siren operating passes by the pitch of the siren appears to increase as the vehicle approaches and decrease as it goes away. This change in pitch is known as the **Doppler effect** This effect occurs in any situation where waves are reflected from a moving object.

A radar gun produces a signal which is bounced off the target moving vehicle so that when it returns to the gun, which also acts as a receiver, it has changed pitch. The speed of the vehicle can be calculated from the change in pitch. The speed indicated on the radar gun is the speed directly towards or away from the gun. However it is not usual to place oneself directly in front of moving vehicle when using the radar gun (Figure 15(a).) Consequently the gun is used at an angle to the line of traffic (Figure 15(b).)

radar
gun

$\underline{v}$

(a)

$\underline{v}$

θ

(b)
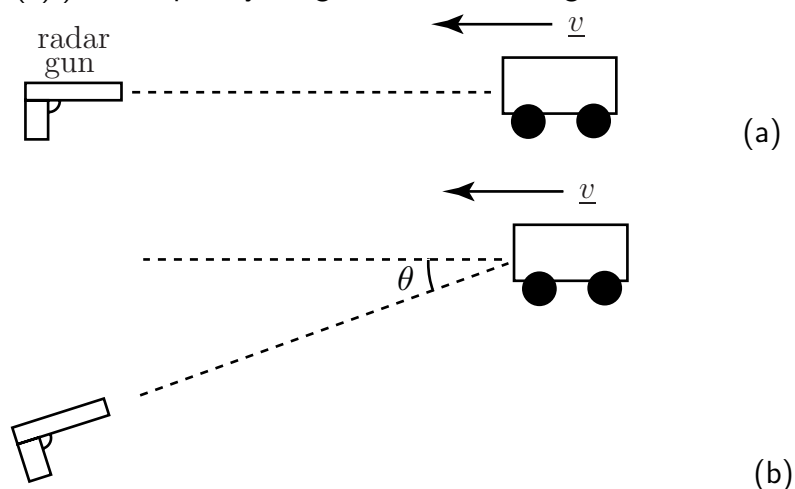
Figure 15

This means that it registers only the component of the velocity towards the gun. Suppose that the true speed along the road is v. Then the component measured by the gun ($v\cos\theta$) is less than v.

Example 2

A safety inspector wishes to check the speed of a train along a straight piece of track. She stands 10 m to the side of the track and uses a radar gun. If the reading on the gun is to be within 5% of the true speed of the train, how far away from the approaching train should the reading be taken?
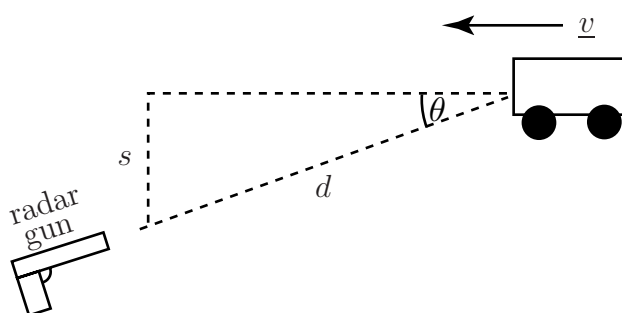
Solution

Figure 16

For an error of 5%, the gun should read $0.95|\underline{v}|$. So

$$|\underline{v}|\cos\theta = 0.95|\underline{v}| \quad \text{or} \quad \cos\theta = 0.95$$

If the distance to the side of the track at which the gun is used is s m and the distance between the radar gun and the train is d m, then

$$\sin\theta = \frac{s}{d}$$

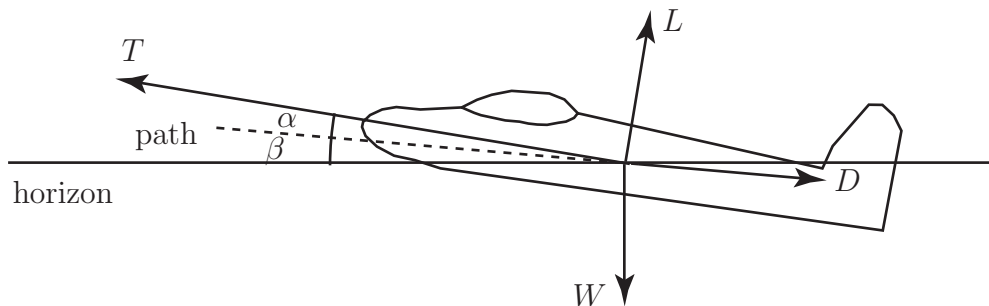Here $s = 10$ so

$$\sqrt{1 - \cos^2\theta} = \frac{10}{d}.$$

So, if $\cos\theta = 0.95$

$$d = \frac{10}{\sqrt{1 - (0.95)^2}} = 32.03$$

This means that the reading should be taken when the train is over 32 m from the radar gun to ensure an error of less than 5%.

The force vectors on an aeroplane in steady flight

The forces acting on an aeroplane are shown in Figure 17.

Figure 17

The **magnitude** (strength) of the forces are indicated by

T: the thrust provided by the engines,

W: the weight,

D: the drag (acting against the direction of flight) and

L: the lift (taken perpendicular to the path.)

In a more realistic situation force vectors in *three* dimensions would need to be considered. These are introduced later in this Workbook.

As the plane is in *steady* flight the sum of the forces in any direction is zero. (If this were not the case, then, by Newton's second law, the non-zero resultant force would cause the aeroplane to accelerate.)

So, resolving forces in the direction of the path:

$$T \cos \alpha - D - W \sin \beta = 0$$

Then, resolving forces perpendicular to the path:

$$T \sin \alpha + L - W \cos \beta = 0$$

If the plane has mass 72 000 tonnes, the drag is 130 kN, the lift is 690 kN and $\beta = 6^o$ find the magnitude of the thrust and the value of α to maintain steady flight. From these two equations we see:

$$T \cos \alpha = D + W \sin \beta = 130000 + (72000)(9.81) \sin 6^o = 203830.54$$

and

$$T \sin \alpha = W \cos \beta - L = (72000)(9.81) \cos 6^o - 690000 = 12450.71$$

hence

$$\tan \alpha = \frac{T \sin \alpha}{T \cos \alpha} = \frac{12450.71}{203830.54} = 0.061084 \quad \rightarrow \quad \alpha = 3.50^0$$

and consequently, for the thrust:

$$T = 204210 \text{ N}.$$

4. Subtraction of vectors

Subtraction of one vector from another is performed by adding the corresponding negative vector. That is, if we seek $\underline{a} - \underline{b}$ we form $\underline{a} + (-\underline{b})$. This is shown geometrically in Figure 18. Note that in the right-hand diagram the arrow on $\underline{b}$ has been reversed to give $-\underline{b}$.
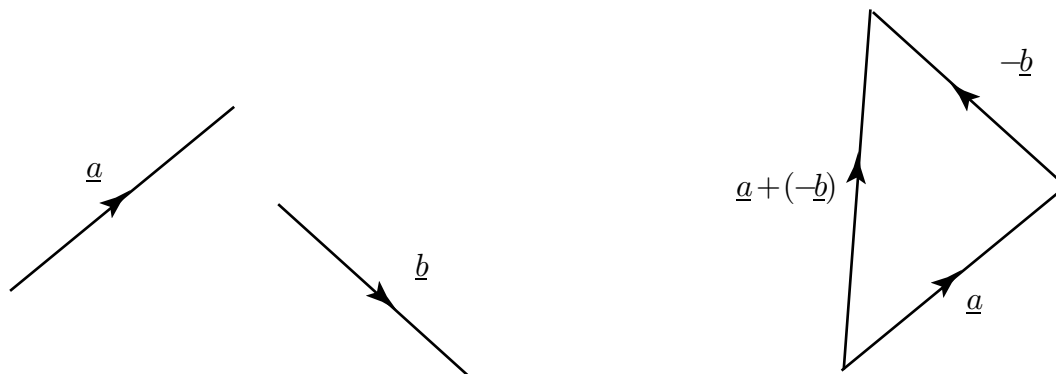
Figure 18: Subtraction of a vector is performed by adding a negative vector

Exercises

1. Vectors $\underline{p}$ and $\underline{q}$ represent two perpendicular sides of a square. Find vector expressions which represent the diagonals of the square.

2. In the rectangle $ABCD$, side AB is represented by the vector $\underline{p}$ and side BC is represented by the vector $\underline{q}$. State the physical significance of the vectors $\underline{p} - \underline{q}$ and $\underline{p} + \underline{q}$.

3. An object is positioned at the origin of a set of axes. Two forces act upon it. The first has magnitude 9 N and acts in the direction of the positive y axis. The second has magnitude 4 N and acts in the direction of the negative x axis. Calculate the magnitude and direction of the resultant force.

4. An object moves in the xy plane with a velocity of 15 m s^{-1} in a direction $48°$ above the positive x axis. Resolve this velocity into two components, one along the x axis and one along the y axis.

Answers

1. $\underline{p} + \underline{q}$, $\underline{q} - \underline{p}$. Acceptable answers are also $-(\underline{p} + \underline{q})$, $\underline{p} - \underline{q}$.

2. $\underline{p} + \underline{q}$ is the diagonal AC, $\underline{p} - \underline{q}$ is the diagonal DB.

3. Magnitude $\sqrt{97}$, at an angle $66°$ above the negative x axis.

4. 10.04 m s^{-1} along the x axis, and 11.15 m s^{-1} along the y axis.

5. Multiplying a vector by a scalar

If k is any positive scalar and $\underline{a}$ is a vector then $k\underline{a}$ is a vector in the same direction as $\underline{a}$ but k times as long. If k is negative, $k\underline{a}$ is a vector in the opposite direction to $\underline{a}$ and k times as long. See Figure 19. The vector $k\underline{a}$ is said to be a **scalar multiple** of $\underline{a}$.

$\underline{a}$ $k\underline{a}$ if k is positive $\underline{a}$ $k\underline{a}$ if k is negative
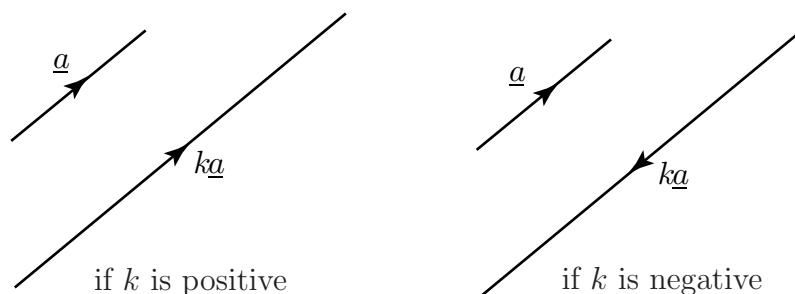
Figure 19: Multiplying a vector by a scalar

The vector $3\underline{a}$ is three times as long as $\underline{a}$ and has the same direction. The vector $\frac{1}{2}\underline{r}$ is in the same direction as $\underline{r}$ but is half as long. The vector $-4\underline{b}$ is in the opposite direction to $\underline{b}$ and four times as long.

For any scalars k and l, and any vectors $\underline{a}$ and $\underline{b}$, the following rules hold:

Key Point 2

$$k(\underline{a} + \underline{b}) = k\underline{a} + k\underline{b}$$
$$(k + l)\underline{a} = k\underline{a} + l\underline{a}$$
$$k(l)\underline{a} = (kl)\underline{a}$$

Task

Using the rules given in Key Point 2, simplify the following:

(a) $3\underline{a} + 7\underline{a}$ (b) $2(7\underline{b})$ (c) $4\underline{q} + 4\underline{r}$
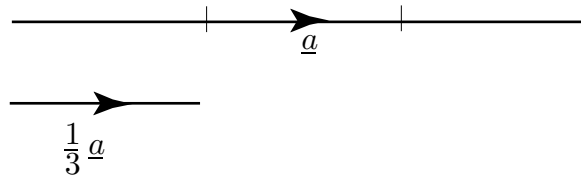
Your solution

Answer

(a) Using the second rule, $3\underline{a} + 7\underline{a}$ can be simplified to $(3 + 7)\underline{a} = 10\underline{a}$.

(b) Using the third rule $2(7\underline{b}) = (2 \times 7)\underline{b} = 14\underline{b}$.

(c) Using the first rule $4\underline{q} + 4\underline{r} = 4(\underline{q} + \underline{r})$.

Unit vectors

A vector which has a magnitude of 1 is called a **unit vector**. If $\underline{a}$ has magnitude 3, then a unit vector in the direction of $\underline{a}$ is $\frac{1}{3}\underline{a}$, as shown in Figure 20.

Figure 20: A unit vector has length one unit

A unit vector in the direction of a given vector is found by dividing the given vector by its magnitude: A unit vector in the direction of $\underline{a}$ is given the 'hat' symbol $\hat{\underline{a}}$.

Key Point 3

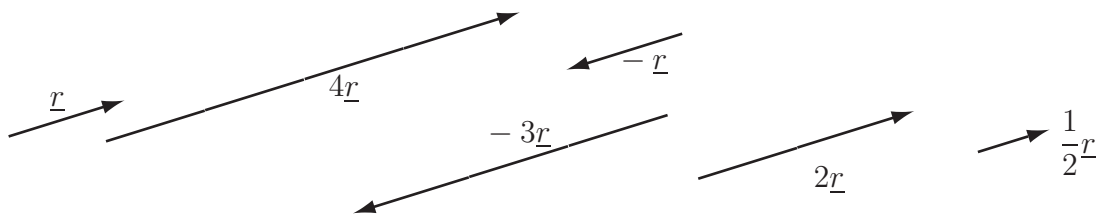A unit vector can be found by dividing a vector by its modulus.

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

Exercises

1. Draw an arbitrary vector $\underline{r}$. On your diagram draw $2\underline{r}$, $4\underline{r}$, $-\underline{r}$, $-3\underline{r}$ and $\frac{1}{2}\underline{r}$.

2. In triangle OAB the point P divides AB internally in the ratio $m:n$. If $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$ depict this on a diagram and then find an expression for $\overrightarrow{OP}$ in terms of $\underline{a}$ and $\underline{b}$.

Answers

1.

2. $\overrightarrow{OP} = \underline{a} + \dfrac{m}{m+n}(\underline{b} - \underline{a}) = \dfrac{n\underline{a} + m\underline{b}}{m+n}$.