Solution by Gauss Elimination

8.3

Introduction

Engineers often need to solve large systems of linear equations; for example in determining the forces in a large framework or finding currents in a complicated electrical circuit. The method of Gauss elimination provides a systematic approach to their solution.

Prerequisites

Before starting this Section you should ...

- be familiar with matrix algebra

Learning Outcomes

On completion you should be able to ...

- identify the row operations which allow the reduction of a system of linear equations to upper triangular form
- use back-substitution to solve a system of equations in echelon form
- understand and use the method of Gauss elimination to solve a system of three simultaneous linear equations

1. Solving three equations in three unknowns

The easiest set of three simultaneous linear equations to solve is of the following type:

$$3x_1 = 6,$$

$$2x_2 = 5,$$

$$4x_3 = 7$$

which obviously has solution $[x_1, x_2, x_3]^T = \left[2, \frac{5}{2}, \frac{7}{4}\right]^T$ or $x_1 = 2, x_2 = \frac{5}{2}, x_3 = \frac{7}{4}$.
In matrix form $AX = B$ the equations are

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$$

where the matrix of coefficients, A, is clearly diagonal.

Task

Solve the equations

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}.$$

Your solution

Answer

$[x_1, x_2, x_3]^T = [4, -2, -2]^T$.

The next easiest system of equations to solve is of the following kind:

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ 2x_2 + x_3 &= 12 \\ 3x_3 &= 6. \end{aligned}$$

The last equation can be solved immediately to give $x_3 = 2$.
Substituting this value of x_3 into the second equation gives

$$2x_2 + 2 = 12 \qquad \text{from which} \qquad 2x_2 = 10 \qquad \text{so that} \qquad x_2 = 5$$

Substituting these values of x_2 and x_3 into the first equation gives

$$3x_1 + 5 - 2 = 0 \qquad \text{from which} \qquad 3x_1 = -3 \qquad \text{so that} \qquad x_1 = -1$$

Hence the solution is $[x_1, x_2, x_3]^T = [-1, 5, 2]^T$.
This process of solution is called **back-substitution**.
In matrix form the system of equations is

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix}.$$

The matrix of coefficients is said to be **upper triangular** because all elements below the leading diagonal are zero. Any system of equations in which the coefficient matrix is triangular (whether upper or lower) will be particularly easy to solve.

Task

Solve the following system of equations by back-substitution.

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}.$$

Write the equations in expanded form:

Your solution

Answer

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 7 \\ 3x_2 - x_3 &= 5 \\ 2x_3 &= 2 \end{aligned}$$

Now find the solution for x_3:

Your solution

$x_3 =$

Answer

The last equation can be solved immediately to give $x_3 = 1$.

Using this value for x_3, obtain x_2 and x_1:

Your solution

$x_2 =$ $\qquad$ $x_1 =$

Answer

$x_2 = 2, \quad x_1 = 3$. Therefore the solution is $x_1 = 3, x_2 = 2$ and $x_3 = 1$.

Although we have worked so far with integers this will not always be the case and fractions will enter the solution process. We must then take care and it is always wise to check that the equations balance using the calculated solution.

2. The general system of three simultaneous linear equations

In the previous subsection we met systems of equations which could be solved by back-substitution alone. In this Section we meet systems which are not so amenable and where preliminary work must be done before back-substitution can be used.

Consider the system

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 14 \\ 2x_1 - x_2 - 3x_3 &= 3 \\ 4x_1 + 5x_2 - x_3 &= 7 \end{aligned}$$

We will use the solution method known as **Gauss elimination**, which has three stages. In the first stage the equations are written in matrix form. In the second stage the matrix equations are replaced by a system of equations having the same solution but which are in **triangular form**. In the final stage the new system is solved by **back-substitution**.

Stage 1: Matrix Formulation

The first step is to write the equations in matrix form:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & -3 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \\ 7 \end{bmatrix}.$$

Then, for conciseness, we combine the matrix of coefficients with the column vector of right-hand sides to produce the **augmented matrix**:

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right]$$

If the general system of equations is written $AX = B$ then the augmented matrix is written $[A|B]$.

Hence the first equation

$$x_1 + 3x_2 + 5x_3 = 14$$

is replaced by the first row of the augmented matrix,

$1 \quad 3 \quad 5 \quad | \quad 14$ and so on.

Stage 1 has now been completed. We will next triangularise the matrix of coefficients by means of **row operations**. There are three possible row operations:

- interchange two rows;
- multiply or divide a row by a non-zero constant factor;
- add to, or subtract from, one row a multiple of another row.

Note that interchanging two rows of the augmented matrix is equivalent to interchanging the two corresponding equations. The shorthand notation we use is introduced by example. To interchange row 1 and row 3 we write $R1 \leftrightarrow R3$. To divide row 2 by 5 we write $R2 \div 5$. To add three times row 1 to row 2, we write $R2 + 3R1$. In the Task which follows you will see where these annotations are placed.

Note that these operations neither create nor destroy solutions so that at every step the system of equations has the same solution as the original system.

Stage 2: Triangularisation

The second stage proceeds by first eliminating x_1 from the second and third equations using row operations.

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right] \begin{array}{l} \\ R2 - 2\times R1 \\ R3 - 4\times R1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right]$$

In the above we have subtracted twice row (equation) 1 from row (equation) 2.
In full these operations would be written, respectively, as

$$(2x_1 - x_2 - 3x_3) - 2(x_1 + 3x_2 + 5x_3) = 3 - 2\times 14 \quad \text{or} \quad -7x_2 - 13x_3 = -25$$

and

$$(4x_1 + 5x_2 - x_3) - 4(x_1 + 3x_2 + 5x_3) = 7 - 4\times 14 \quad \text{or} \quad -7x_2 - 21x_3 = -49.$$

Now since all the elements in rows 2 and 3 are negative we multiply throughout by -1:

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right] \begin{array}{l} \\ R2 \times (-1) \\ R3 \times (-1) \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 7 & 21 & 49 \end{array}\right]$$

Finally, we eliminate x_2 from the third equation by subtracting equation 2 from equation 3
i.e. $R3 - R2$:

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 7 & 21 & 49 \end{array}\right] \begin{array}{l} \\ \\ R3 - R2 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 0 & 8 & 24 \end{array}\right]$$

The system is now in triangular form.

Stage 3: Back Substitution

Here we solve the equations from bottom to top. At each step of the back substitution process we encounter equations which only have a **single** unknown and so can be easily solved.

Task

Now complete the solution to the above system by back-substitution.

Your solution

Answer
In full the equations are

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 14 \\ 7x_2 + 13x_3 &= 25 \\ 8x_3 &= 24 \end{aligned}$$

From the last equation we see that $x_3 = 3$.
Substituting this value into the second equation gives

$$7x_2 + 39 = 25 \quad \text{or} \quad 7x_2 = -14 \quad \text{so that} \quad x_2 = -2.$$

Finally, using these values for x_2 and x_3 in equation 1 gives $x_1 - 6 + 15 = 14$. Hence $x_1 = 5$. The solution is therefore $[x_1, x_2, x_3]^T = [5, -2, 3]^T$

Check that these values satisfy the original system of equations.

Task

Solve

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 2 \\ 4x_1 + x_2 + 2x_3 &= 2 \\ x_1 - x_2 + 3x_3 &= 3 \end{aligned}$$

Write down the augmented matrix for this system and then interchange rows 1 and 3:

Your solution

Answer
Augmented matrix

$$\left[\begin{array}{rrr|r} 2 & -3 & 4 & 2 \\ 4 & 1 & 2 & 2 \\ 1 & -1 & 3 & 3 \end{array}\right] \quad R1 \leftrightarrow R3 \quad \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 4 & 1 & 2 & 2 \\ 2 & -3 & 4 & 2 \end{array}\right]$$

Now subtract suitable multiples of row 1 from row 2 and from row 3 to eliminate the x_1 coefficient from rows 2 and 3:

Your solution

Answer

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 4 & 1 & 2 & 2 \\ 2 & -3 & 4 & 2 \end{array}\right] \quad \begin{array}{l} \\ R2 - 4R1 \\ R3 - 2R1 \end{array} \quad \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 5 & -10 & -10 \\ 0 & -1 & -2 & -4 \end{array}\right]$$

Now divide row 2 by 5 and add a suitable multiple of the result to row 3:

Your solution

Answer

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 5 & -10 & -10 \\ 0 & -1 & -2 & -4 \end{array}\right] \quad R2 \div 5 \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -2 & -4 \end{array}\right] \quad \begin{array}{r} \\ \Rightarrow \\ R3 + R2 \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -4 & -6 \end{array}\right]$$

Now complete the solution using back-substitution:

Your solution

Answer

The equations in full are

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 3 \\ x_2 - 2x_3 &= -2 \\ -4x_3 &= -6. \end{aligned}$$

The last equation reduces to $x_3 = \frac{3}{2}$.

Using this value in the second equation gives $x_2 - 3 = -2$ so that $x_2 = 1$.

Finally, $x_1 - 1 + \frac{9}{2} = 3$ so that $x_1 = -\frac{1}{2}$.

The solution is therefore $[x_1, x_2, x_3]^T = \left[-\frac{1}{2}, 1, \frac{3}{2}\right]^T$.

You should check these values in the original equations to ensure that the equations balance.
Again we emphasise that we chose a particular set of procedures in Stage 1. This was chosen mainly to keep the arithmetic simple by delaying the introduction of fractions. Sometimes we are courageous and take fewer, harder steps.

An important point to note is that when in Stage 2 we wrote $R2 - 2R1$ against row 2; what we meant is that row 2 is replaced by the combination (row $2) - 2\times$(row 1).
In general, the operation

$$\text{row } i - \alpha \times \text{row } j$$

means replace **row** i by the combination

$$\text{row } i - \alpha \times \text{row } j.$$

3. Equations which have an infinite number of solutions

Consider the following system of equations

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 3 \\ 2x_1 - 3x_2 + 4x_3 &= -4 \\ x_1 - x_2 + x_3 &= -1 \end{aligned}$$

In augmented form we have:

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 2 & -3 & 4 & -4 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

Now performing the usual Gauss elimination operations we have

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 2 & -3 & 4 & -4 \\ 1 & -1 & 1 & -1 \end{array}\right] \begin{array}{c} \\ R2 - 2 \times R1 \\ R3 - R1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 0 & -5 & 10 & -10 \\ 0 & -2 & 4 & -4 \end{array}\right]$$

Now applying $R2 \div -5$ and $R3 \div -2$ gives

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \end{array}\right]$$

Then $R2 - R3$ gives

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that all the elements in the last row are zero. This means that the variable x_3 can take any value whatsoever, so let $x_3 = t$ then using back substitution the second row now implies

$$x_2 = 2 + 2x_3 = 2 + 2t$$

and then the first row implies

$$x_1 = 3 - x_2 + 3x_3 = 3 - (2 + 2t) + 3(t) = 1 + t$$

In this example the system of equations has an infinite number of solutions:

$$x_1 = 1 + t, \qquad x_2 = 2 + 2t, \qquad x_3 = t \quad \text{or} \quad [x_1, x_2, x_3]^T = [1 + t, 2 + 2t, t]^T$$

where t can be assigned any value. For every value of t these expressions for x_1, x_2 and x_3 will simultaneously satisfy each of the three given equations.

Systems of linear equations arise in the modelling of electrical circuits or networks. By breaking down a complicated system into simple loops, Kirchhoff's law can be applied. This leads to a set of linear equations in the unknown quantities (usually currents) which can easily be solved by one of the methods described in this Workbook.

Engineering Example 3

Currents in three loops

In the circuit shown find the currents (i_1, i_2, i_3) in the loops.

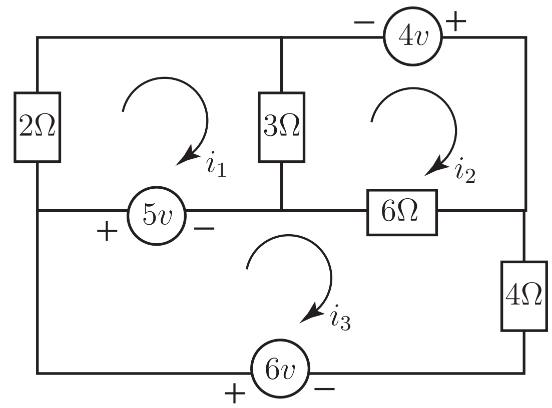

Figure 2

Solution

Loop 1 gives

$$2(i_1) + 3(i_1 - i_2) = 5 \quad \rightarrow \quad 5i_1 - 3i_2 = 5$$

Loop 2 gives

$$6(i_2 - i_3) + 3(i_2 - i_1) = 4 \quad \rightarrow \quad -3i_1 + 9i_2 - 6i_3 = 4$$

Loop 3 gives

$$6(i_3 - i_2) + 4(i_3) = 6 - 5 \quad \rightarrow \quad -6i_2 + 10i_3 = 1$$

Note that in loop 3, the current generated by the $6v$ cell is positive and for the $5v$ cell negative in the direction of the arrow.
In matrix form

$$\begin{bmatrix} 5 & -3 & 0 \\ -3 & 9 & -6 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

Solving gives

$$i_1 = \frac{34}{15}, \quad i_2 = \frac{19}{9}, \quad i_3 = \frac{41}{30}$$

Engineering Example 4

Velocity of a rocket

The upward velocity of a rocket, measured at 3 different times, is shown in the following table

Time, t (seconds)	Velocity, v (metres/second)
5	106.8
8	177.2
12	279.2

The velocity over the time interval $5 \leq t \leq 12$ is approximated by a quadratic expression as

$$v(t) = a_1t^2 + a_2t + a_3$$

Find the values of a_1, a_2 and a_3.

Solution

Substituting the values from the table into the quadratic equation for $v(t)$ gives:

$$\begin{aligned} 106.8 &= 25a_1 + 5a_2 + a_3 \\ 177.2 &= 64a_1 + 8a_2 + a_3 \\ 279.2 &= 144a_1 + 12a_2 + a_3 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Applying one of the methods from this Workbook gives the solution as

$$a_1 = 0.2905 \qquad a_2 = 19.6905 \qquad a_3 = 1.0857 \quad \text{to 4 d.p.}$$

As the original values were all **experimental observations** then the values of the unknowns are all **approximations**. The relation $v(t) = 0.2905t^2 + 19.6905t + 1.0857$ can now be used to predict the approximate position of the rocket for any time within the interval $5 \leq t \leq 12$.

Exercises

Solve the following using Gauss elimination:

1.
$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & 0 \end{array}$$

2.
$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 1 \\ -x_1 & & & + & x_3 & = & 1 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$$

3.
$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 3 \\ x_1 & - & 2x_2 & - & x_3 & = & 1 \end{array}$$

4.
$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & -1 \\ 3x_1 & + & x_2 & + & x_3 & = & 4 \\ 11x_1 & - & x_2 & - & 3x_3 & = & 10 \end{array}$$

You may need to think carefully about this system.

Answers

(1) $x_1 = \dfrac{8}{3},\ x_2 = -4,\ x_3 = \dfrac{4}{3}$

(2) $x_1 = \dfrac{1}{2},\ x_2 = 1,\ x_3 = \dfrac{3}{2}$

(3) $x_1 = 2,\ x_2 = 1,\ x_3 = -1$

(4) infinite number of solutions: $x_1 = t,\ x_2 = 11 - 10t,\ x_3 = 7t - 7$