Solution by Inverse Matrix Method 8.2

Introduction

The power of matrix algebra is seen in the representation of a system of simultaneous linear equations as a matrix equation. Matrix algebra allows us to write the solution of the system using the inverse matrix of the coefficients. In practice the method is suitable only for small systems. Its main use is the theoretical insight into such problems which it provides.

Prerequisites

Before starting this Section you should . . .

- be familiar with the basic rules of matrix algebra
- be able to evaluate 2×2 and 3×3 determinants
- be able to find the inverse of 2×2 and 3×3 matrices

Learning Outcomes

On completion you should be able to . . .

- use the inverse matrix of coefficients to solve a system of two linear simultaneous equations
- use the inverse matrix of coefficients to solve a system of three linear simultaneous equations
- recognise and identify cases where there is no solution or no unique solution

1. Solving a system of two equations using the inverse matrix

If we have one linear equation

$$ax = b$$

in which the unknown is x and a and b are constants and $a \neq 0$ then $x = \dfrac{b}{a} = a^{-1}b$.
What happens if we have more than one equation and more than one unknown? In this Section we copy the algebraic solution $x = a^{-1}b$ used for a single equation to solve a system of linear equations. As we shall see, this will be a very natural way of solving the system if it is first written in matrix form.
Consider the system

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 - 2x_2 &= -1. \end{aligned}$$

In matrix form this becomes

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \qquad \text{which is of the form} \qquad AX = B.$$

If A^{-1} exists then the solution is

$$X = A^{-1}B. \qquad \text{(compare the solution } x = a^{-1}b \text{ above)}$$

Task

Given the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ find its determinant. What does this tell you about A^{-1}?

Your solution

Answer
$|A| = 2 \times (-2) - 1 \times 3 = -7$
since $|A| \neq 0$ then A^{-1} exists.

Now find A^{-1}

Your solution

Answer
$$A^{-1} = \frac{1}{(-7)} \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

Task

Solve the system $AX = B$ where $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ and B is $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

Your solution

Answer

$X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence $x_1 = 1,\ x_2 = 1.$

Task

Use the inverse matrix method to solve

$$2x_1 + 3x_2 = 3$$
$$5x_1 + 4x_2 = 11$$

Your solution

Answer

$AX = B$ is $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$

$$|A| = 2\times 4 - 3\times 5 = -7 \text{ and } A^{-1} = -\frac{1}{7}\begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix}$$

Using $X = A^{-1}B$:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 11 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} 4\times 3 - 3\times 11 \\ -5\times 3 + 2\times 11 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -21 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

So $x_1 = 3,\ x_2 = -1$

Engineering Example 2

Currents in two loops

In the circuit shown find the currents $(i_1,\ i_2)$ in the loops.

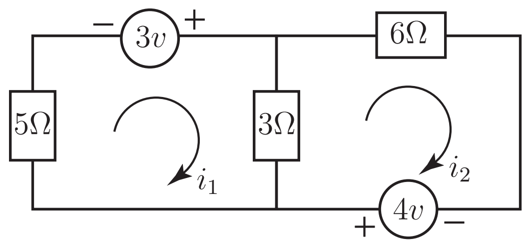

Figure 1

Solution

We note that the current across the $3\ \Omega$ resistor (travelling top to bottom in the diagram) is given by $(i_1 - i_2)$. With this proviso we can apply Kirchhoff's law:

In the left-hand loop $\quad 3(i_1 - i_2) + 5i_1 = 3 \quad \rightarrow \quad 8i_1 - 3i_2 = 3$

In the right-hand loop $\quad 3(i_2 - i_1) + 6i_2 = 4 \quad \rightarrow \quad -3i_1 + 9i_2 = 4$

In matrix form:

$$\begin{bmatrix} 8 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 8 & -3 \\ -3 & 9 \end{bmatrix}$ is $\dfrac{1}{63}\begin{bmatrix} 9 & 3 \\ 3 & 8 \end{bmatrix}$ so solving gives

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \frac{1}{63}\begin{bmatrix} 9 & 3 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{63}\begin{bmatrix} 39 \\ 41 \end{bmatrix}$$

so $\quad i_1 = \dfrac{39}{63} \quad i_2 = \dfrac{41}{63}$

2. Non-unique solutions

The key to obtaining a unique solution of the system $AX = B$ is to find A^{-1}. What happens when A^{-1} does not exist?
Consider the system

$$2x_1 + 3x_2 = 5 \qquad (1)$$

$$4x_1 + 6x_2 = 10 \qquad (2)$$

In matrix form this becomes

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

Task

Identify the matrix A and hence find A^{-1}.

Your solution

Answer

$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ and $|A| = 2 \times 6 - 4 \times 3 = 0$. Hence A^{-1} does not exist.

Looking at the original system we see that Equation (2) is simply Equation (1) with all coefficients doubled. In effect we have only one equation for the two unknowns x_1 and x_2. The equations are **consistent** and have **infinitely many solutions**.

If we let x_2 assume a **particular** value, t say, then x_1 must take the value $x_1 = \dfrac{1}{2}(5 - 3t)$ i.e. the solution to the given equations is:

$$x_2 = t, \quad x_1 = \frac{1}{2}(5 - 3t), \qquad \text{where } t \text{ is called a parameter.}$$

For each value of t there are unique values for x_1 and x_2 which satisfy the original system of equations. For example, if $t = 1$, then $x_2 = 1$, $x_1 = 1$, if $t = -3$ then $x_2 = -3$, $x_1 = 7$ and so on.
Now consider the system

$$2x_1 + 3x_2 = 5 \qquad (3)$$

$$4x_1 + 6x_2 = 9 \qquad (4)$$

Since the left-hand sides are the same as those in the previous system then A is the same and again A^{-1} does not exist. There is **no solution** to the Equations (3) and (4).
However, if we double Equation (3) we obtain

$$4x_1 + 6x_2 = 10,$$

which conflicts with Equation (4). There are thus no solutions to (3) and (4) and the equations are said to be **inconsistent**.

Task

What can you conclude about the solutions of the following systems?

(a) $\begin{array}{rcrcl} x_1 & - & 2x_2 & = & 1 \\ 3x_1 & - & 6x_2 & = & 3 \end{array}$ (b) $\begin{array}{rcrcl} 3x_1 & + & 2x_2 & = & 7 \\ -6x_1 & - & 4x_2 & = & 5 \end{array}$

First write the systems in matrix form and find $|A|$:

Your solution

(a)

(b)

Answer

(a) $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $|A| = -6 + 6 = 0;$

(b) $\begin{bmatrix} 3 & 2 \\ -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ $|A| = -12 + 12 = 0.$

Now compare the two equations in each system in turn:

Your solution

(a)

(b)

Answer

(a) The second equation is 3 times the first equation. There are infinitely many solutions of the form $x_2 = t,\ x_1 = 1 + 2t$ where t is arbitrary.

(b) If we multiply the first equation by (-2) we obtain $-6x_1 - 4x_2 = -14$ which is in conflict with the second equation. The equations are inconsistent and have no solution.

3. Solving three equations in three unknowns

It is much more tedious to use the inverse matrix to solve a system of three equations although in principle, the method is the same as for two equations.
Consider the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 5 \\ 3x_1 - x_2 + 2x_3 &= 12 \end{aligned}$$

We met this system in Section 8.1 where we found that $|A| = 10$. Hence A^{-1} exists.

Task

Find A^{-1} by the method of determinants.

First form the matrix where each element of A is replaced by its minor:

Your solution

Answer

$$\begin{bmatrix} \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 7 & -5 \\ -3 & -1 & 5 \\ 1 & -3 & 5 \end{bmatrix}.$$

Now use the 3×3 array of signs to obtain the matrix of cofactors:

Your solution

Answer

The array of signs is $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ so that we obtain $\begin{bmatrix} 1 & -7 & -5 \\ 3 & -1 & -5 \\ 1 & 3 & 5 \end{bmatrix}$.

Now transpose this matrix and divide by $|A|$ to obtain A^{-1}:

Your solution

Answer

Transposing gives $\begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$. Finally, $A^{-1} = \dfrac{1}{10}\begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$.

Now use $X = A^{-1}B$ to solve the system of linear equations:

Your solution

Answer

$$X = \frac{1}{10}\begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 30 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$ Then $x_1 = 3,\ x_2 = 1,\ x_3 = 2.$

Comparing this approach to the use of Cramer's rule for three equations (in subsection 2 of Section 8.1) we can say that the two methods are both rather tedious!

Equations with no unique solution

If $|A| = 0$, A^{-1} does not exist and therefore it is easy to see that the system of equations has no unique solution. But it is not obvious whether this is because the equations are inconsistent and have no solution or whether they are consistent and have infinitely many solutions.

Task

Consider the systems

(a) $$\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 - 2x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= 7 \end{aligned}$$

(b) $$\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 - 2x_3 &= 3 \\ x_1 - 11x_2 + 9x_3 &= 13 \end{aligned}$$

In system (a) add the first equation to the second. What does this tell you about the system?

Your solution

Answer

The sum is $3x_1 + 2x_2 - x_3 = 7$, which is identical to the third equation. Thus, essentially, there are only two equations $x_1 - x_2 + x_3 = 4$ and $3x_1 + 2x_2 - x_3 = 7$. Now adding these two gives $4x_1 + x_2 = 11$ or $x_2 = 11 - 4x_1$ and then

$$x_3 = 4 - x_1 + x_2 = 4 - x_1 + 11 - 4x_1 = 15 - 5x_1$$

Hence if we give x_1 a value, t say, then $x_2 = 11 - 4t$ and $x_3 = 15 - 5t$. Thus there is an infinite number of solutions (one for each value of t).

In system (b) take the combination 5 times the first equation minus 2 times the second equation. What does this tell you about the system?

Your solution

Answer
The combination is $x_1 - 11x_2 + 9x_3 = 14$, which conflicts with the third equation. There is thus no solution.

In practice, systems containing three or more linear equations are best solved by the method which we shall introduce in Section 8.3.

Exercises

1. Solve the following using the inverse matrix method:

(a) $\begin{array}{rcrcl} 2x & - & 3y & = & 1 \\ 4x & + & 4y & = & 2 \end{array}$ (b) $\begin{array}{rcrcl} 2x & - & 5y & = & 2 \\ -4x & + & 10y & = & 1 \end{array}$ (c) $\begin{array}{rcrcl} 6x & - & y & = & 0 \\ 2x & - & 4y & = & 1 \end{array}$

2. Solve the following equations using matrix methods:

(a) $\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & 0 \end{array}$ (b) $\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 1 \\ -x_1 & & & + & x_3 & = & 1 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$

Answers

1. (a) $x = \dfrac{1}{2},\ y = 0$ (b) A^{-1} does not exist. (c) $x = -\dfrac{1}{22},\ y = -\dfrac{3}{11}$

2. (a) $x_1 = \dfrac{8}{3},\ x_2 = -4,\ x_3 = \dfrac{4}{3}$ (b) $x_1 = \dfrac{1}{2},\ x_2 = 1,\ x_3 = \dfrac{3}{2}$