Contents 8

Matrix Solution of Equations

Learning outcomes

In this Workbook you will learn to apply your knowledge of matrices to solve systems of linear equations. Such systems of equations arise very often in mathematics, science and engineering. Three basic techniques are outlined, Cramer's method, the inverse matrix approach and the Gauss elimination method. The Gauss elimination method is, by far, the most widely used (since it can be applied to all systems of linear equations). However, you will learn that, for certain (usually small) systems of linear equations the other two techniques may be better.

Solution by Cramer's Rule

8.1

Introduction

The need to solve systems of linear equations arises frequently in engineering. The analysis of electric circuits and the control of systems are two examples. Cramer's rule for solving such systems involves the calculation of determinants and their ratio. For systems containing only a few equations it is a useful method of solution.

Prerequisites

Before starting this Section you should ...

- be able to evaluate 2×2 and 3×3 determinants

Learning Outcomes

On completion you should be able to ...

- state and apply Cramer's rule to find the solution of two simultaneous linear equations
- state and apply Cramer's rule to find the solution of three simultaneous linear equations
- recognise cases where the solution is not unique or a solution does not exist

1. Solving two equations in two unknowns

If we have one linear equation

$$ax = b$$

in which the unknown is x and a and b are constants then there are just three possibilities:

- $a \neq 0$ then $x = \dfrac{b}{a} = a^{-1}b$. In this case the equation $ax = b$ has a **unique solution** for x.
- $a = 0$, $b = 0$ then the equation $ax = b$ becomes $0 = 0$ and any value of x will do. In this case the equation $ax = b$ has **infinitely many solutions**.
- $a = 0$ and $b \neq 0$ then $ax = b$ becomes $0 = b$ which is a contradiction. In this case the equation $ax = b$ has **no solution** for x.

What happens if we have more than one equation and more than one unknown? We shall find that the solutions to such systems can be characterised in a manner similar to that occurring for a single equation; that is, a system may have a unique solution, an infinity of solutions or no solution at all. In this Section we examine a method, known as Cramer's rule and employing determinants, for solving systems of linear equations.
Consider the equations

$$ax + by = e \qquad (1)$$

$$cx + dy = f \qquad (2)$$

where a, b, c, d, e, f are given numbers. The variables x and y are unknowns we wish to find. The pairs of values of x and y which **simultaneously** satisfy both equations are called solutions. Simple algebra will eliminate the variable y between these equations. We multiply equation (1) by d, equation (2) by b and subtract:

first, $(1) \times d \qquad adx + bdy = ed$

then, $(2) \times b \qquad bcx + bdy = bf$

(we multiplied in this way to make the coefficients of y equal.)
Now subtract to obtain

$$(ad - bc)x = ed - bf \qquad (3)$$

Task

Starting with equations (1) and (2) above, eliminate x.

Your solution

Answer

Multiply equation (1) by c and equation (2) by a to obtain

$$acx + bcy = ec \qquad \text{and} \qquad acx + ady = af.$$

Now subtract to obtain

$$(bc - ad)y = ec - af$$

If we multiply this last equation in the Task above by -1 we obtain

$$(ad - bc)y = af - ec \tag{4}$$

Dividing equations (3) and (4) by $ad - bc$ we obtain the solutions

$$x = \frac{ed - bf}{ad - bc}\ , \quad y = \frac{af - ec}{ad - bc} \tag{5}$$

There is of course one proviso: if $ad - bc = 0$ then neither x nor y has a defined value.

If we choose to express these solutions in terms of determinants we have the formulation for the solution of simultaneous equations known as **Cramer's rule**.

If we define Δ as the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and provided $\Delta \neq 0$ then the **unique** solution of the equations

$$ax + by = e$$

$$cx + dy = f$$

is by (5) given by

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta} \quad \text{where} \quad \Delta_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

Now Δ is the determinant of coefficients on the left-hand sides of the equations. In the expression Δ_x the coefficients of x (i.e. $\begin{pmatrix} a \\ c \end{pmatrix}$ which is column 1 of Δ) are replaced by the terms on the right-hand sides of the equations (i.e. by $\begin{pmatrix} e \\ f \end{pmatrix}$). Similarly in Δ_y the coefficients of y (column 2 of Δ) are replaced by the terms on the right-hand sides of the equations.

Key Point 1

Cramer's Rule for Two Equations

The unique solution to the equations:

$$ax + by = e$$
$$cx + dy = f$$

is given by:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}$$

in which

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \Delta_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}, \qquad \Delta_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

If $\Delta = 0$ this method of solution cannot be used.

Task

Use Cramer's rule to solve the simultaneous equations

$$\begin{aligned} 2x + y &= 7 \\ 3x - 4y &= 5 \end{aligned}$$

Your solution

Answer

Calculating $\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11$. Since $\Delta \neq 0$ we can proceed with Cramer's solution.

$$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11 \qquad x = \frac{1}{\Delta}\begin{vmatrix} 7 & 1 \\ 5 & -4 \end{vmatrix}, \quad y = \frac{1}{\Delta}\begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix}$$

i.e. $x = \dfrac{(-28-5)}{(-11)}$, $y = \dfrac{(10-21)}{(-11)}$ implying: $x = \dfrac{-33}{-11} = 3$, $y = \dfrac{-11}{-11} = 1.$

You can check by direct substitution that these are the exact solutions to the equations.

Task

Use Cramer's rule to solve the equations

$$\text{(a)}\quad \begin{aligned} 2x-3y &= 6\\ 4x-6y &= 12 \end{aligned} \qquad\qquad \text{(b)}\quad \begin{aligned} 2x-3y &= 6\\ 4x-6y &= 10 \end{aligned}$$

Your solution

Answer

You should have checked $\begin{vmatrix} 2 & -3\\ 4 & -6 \end{vmatrix}$ first, since

$\begin{vmatrix} 2 & -3\\ 4 & -6 \end{vmatrix} = -12-(-12) = 0$. Hence there is no unique solution in either case.

In the system (a) the second equation is twice the first so there are infinitely many solutions. (Here we can give y *any* value we wish, t say; but then the x value is always $(6+3t)/2$. So for each value of t there are values for x and y which simultaneously satisfy both equations. There is an infinite number of possible solutions). In (b) the equations are inconsistent (since the first is $2x-3y=6$ and the second is $2x-3y=5$ which is not possible). Hence there are no solutions.

Notation

For ease of generalisation to larger systems we write the two-equation system in a different notation:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1\\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Here the unknowns are x_1 and x_2, the right-hand sides are b_1 and b_2 and the coefficients are a_{ij} where, for example, a_{21} is the coefficient of x_1 in equation two. In general, a_{ij} is the coefficient of x_j in equation i.

Cramer's rule can then be stated as follows:

If $\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{vmatrix} \neq 0$, then the equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1\\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

have solution

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12}\\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{vmatrix}}\,, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1\\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{vmatrix}}.$$

2. Solving three equations in three unknowns

Cramer's rule can be extended to larger systems of simultaneous equations but the calculational effort increases rapidly as the size of the system increases. We quote Cramer's rule for a system of three equations.

Key Point 2

Cramer's Rule for Three Equations

The unique solution to the system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

is

$$x_1 = \frac{\Delta_{x_1}}{\Delta}, \quad x_2 = \frac{\Delta_{x_2}}{\Delta}, \quad x_3 = \frac{\Delta_{x_3}}{\Delta}$$

in which

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and

$$\Delta_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad \Delta_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \qquad \Delta_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

If $\Delta = 0$ this method of solution cannot be used.

Notice that the structure of the fractions is similar to that for the two-equation case. For example, the determinant forming the numerator of x_1 is obtained from the determinant of coefficients, Δ, by replacing the first column by the right-hand sides of the equations.

Notice too the increase in calculation: in the two-equation case we had to evaluate three 2×2 determinants, whereas in the three-equation case we have to evaluate four 3×3 determinants. Hence Cramer's rule is not really practicable for larger systems.

Task

Use Cramer's rule to solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 5 \\ 3x_1 - x_2 + 2x_3 &= 12. \end{aligned}$$

First check that $\Delta \neq 0$:

Your solution

Answer

$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix}.$

Expanding along the top row,

$$\begin{aligned} \Delta &= 1 \times \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ &= 1 \times (2-1) + 2 \times (4+3) + 1 \times (-2-3) \\ &= 1 + 14 - 5 = 10 \end{aligned}$$

Now find the value of x_1. First write down the expression for x_1 in terms of determinants:

Your solution

Answer

$x_1 = \begin{vmatrix} 3 & -2 & 1 \\ 5 & 1 & -1 \\ 12 & -1 & 2 \end{vmatrix} \div \Delta$

Now calculate x_1 explicitly:

Your solution

Answer

The numerator is found by expanding along the top row to be

$$\begin{aligned}3 \quad &\times \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) \times \begin{vmatrix} 5 & -1 \\ 12 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 5 & 1 \\ 12 & -1 \end{vmatrix} \\ &= 3 \times 1 + 2 \times 22 + 1 \times (-17) \\ &= 30\end{aligned}$$

Hence $x_1 = \dfrac{1}{10} \times 30 = 3$

In a similar way find the values of x_2 and x_3:

Your solution

Answer

$$\begin{aligned} x_2 &= \frac{1}{10}\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & -1 \\ 3 & 12 & 2 \end{vmatrix} \\ &= \frac{1}{10}\left\{1 \times \begin{vmatrix} 5 & -1 \\ 12 & 2 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 3 & 12 \end{vmatrix}\right\} \\ &= \frac{1}{10}\{22 - 3 \times 7 + 9\} = 1 \\ x_3 &= \frac{1}{10}\left\{1 \times \begin{vmatrix} 1 & 5 \\ -1 & 12 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & 5 \\ 3 & 12 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}\right\} \\ &= \frac{1}{10}\{17 + 2 \times 9 + 3 \times (-5)\} = 2 \end{aligned}$$

Engineering Example 1

Stresses and strains on a section of material

Introduction

An important engineering problem is to determine the effect on materials of different types of loading. One way of measuring the effects is through the strain or fractional change in dimensions in the material which can be measured using a strain gauge.

Problem in words

In a homogeneous, isotropic and linearly elastic material, the strains (i.e. fractional displacements) on a section of the material, represented by $\varepsilon_x,\ \varepsilon_y,\ \varepsilon_z$ for the x-, y-, z-directions respectively, can be related to the stresses (i.e. force per unit area), $\sigma_x,\ \sigma_y,\ \sigma_z$ by the following system of equations.

$$\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\ \sigma_x \quad -v\sigma_y \quad -v\sigma_z\ )\\
\varepsilon_y &= \frac{1}{E}(\ -v\sigma_x \quad +\sigma_y \quad -v\sigma_z\ )\\
\varepsilon_z &= \frac{1}{E}(\ -v\sigma_x \quad -v\sigma_y \quad +\sigma_z\ )
\end{aligned}$$

where E is the modulus of elasticity (also called Young's modulus) and v is Poisson's ratio which relates the lateral strain to the axial strain.

Find expressions for the stresses $\sigma_x,\ \sigma_y,\ \sigma_z$, in terms of the strains $\varepsilon_x,\ \varepsilon_y$, and ε_z.

Mathematical statement of problem

The given system of equations can be written as a matrix equation:

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -v & -v \\ -v & 1 & -v \\ -v & -v & 1 \end{pmatrix}\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

We can write this equation as

$$\varepsilon = \frac{1}{E}\boldsymbol{A}\boldsymbol{\sigma}$$

where $\varepsilon = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} 1 & -v & -v \\ -v & 1 & -v \\ -v & -v & 1 \end{pmatrix}$ and $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$

This matrix equation must be solved to find the vector $\boldsymbol{\sigma}$ in terms of the vector ε and the inverse of the matrix $\boldsymbol{A}$.

Mathematical analysis

$$\varepsilon = \frac{1}{E}\boldsymbol{A\sigma}$$

Multiplying both sides of the expression by E we get

$$E\varepsilon = \boldsymbol{A\sigma}$$

Multiplying both sides by $\boldsymbol{A}^{-1}$ we find that:

$$\boldsymbol{A}^{-1}E\varepsilon = \boldsymbol{A}^{-1}\boldsymbol{A\sigma}$$

But $\boldsymbol{A}^{-1}\boldsymbol{A} = \mathbf{I}$ so this becomes

$$\boldsymbol{\sigma} = E\boldsymbol{A}^{-1}\varepsilon$$

To find expressions for the stresses $\sigma_x,\ \sigma_y,\ \sigma_z$, in terms of the strains $\varepsilon_x,\ \varepsilon_y$ and ε_z, we must find the inverse of the matrix **A**.

To find the inverse of $\begin{pmatrix} 1 & -v & -v \\ -v & 1 & -v \\ -v & -v & 1 \end{pmatrix}$ we first find the matrix of minors which is:

$$\begin{pmatrix} \begin{vmatrix} 1 & -v \\ -v & 1 \end{vmatrix} & \begin{vmatrix} -v & -v \\ -v & 1 \end{vmatrix} & \begin{vmatrix} -v & 1 \\ -v & -v \end{vmatrix} \\ \\ \begin{vmatrix} -v & -v \\ -v & 1 \end{vmatrix} & \begin{vmatrix} 1 & -v \\ -v & 1 \end{vmatrix} & \begin{vmatrix} 1 & -v \\ -v & -v \end{vmatrix} \\ \\ \begin{vmatrix} -v & -v \\ 1 & -v \end{vmatrix} & \begin{vmatrix} 1 & -v \\ -v & -v \end{vmatrix} & \begin{vmatrix} 1 & -v \\ -v & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1-v^2 & -v-v^2 & v^2+v \\ -v-v^2 & 1-v^2 & -v-v^2 \\ v^2+v & -v-v^2 & 1-v^2 \end{pmatrix}.$$

We then apply the pattern of signs:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

to obtain the matrix of cofactors

$$\begin{pmatrix} 1-v^2 & v+v^2 & v^2+v \\ v+v^2 & 1-v^2 & v+v^2 \\ v^2+v & v+v^2 & 1-v^2 \end{pmatrix}.$$

To find the adjoint we take the transpose of the above, (which is the same as the original matrix since the matrix is symmetric)

$$\begin{pmatrix} 1-v^2 & v+v^2 & v^2+v \\ v+v^2 & 1-v^2 & v+v^2 \\ v^2+v & v+v^2 & 1-v^2 \end{pmatrix}.$$

The determinant of the original matrix is

$$1 \times (1-v^2) - v(v+v^2) - v(v^2+v) = 1 - 3v^2 - 2v^3.$$

Finally we divide the adjoint by the determinant to find the inverse, giving

$$\frac{1}{1-3v^2-2v^3}\begin{pmatrix} 1-v^2 & v+v^2 & v+v^2 \\ v+v^2 & 1-v^2 & v+v^2 \\ v+v^2 & v+v^2 & 1-v^2 \end{pmatrix}.$$

Now we found that $\boldsymbol{\sigma} = E\boldsymbol{A}^{-1}\varepsilon$ so $\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} = \dfrac{E}{1-3v^2-2v^3}\begin{pmatrix} 1-v^2 & v+v^2 & v+v^2 \\ v+v^2 & 1-v^2 & v+v^2 \\ v+v^2 & v+v^2 & 1-v^2 \end{pmatrix}\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix}$

We can write this matrix equation as 3 equations relating the stresses $\sigma_x,\ \sigma_y,\ \sigma_z,$ in terms of the strains $\varepsilon_x,\ \varepsilon_y$ and ε_z, by multiplying out this matrix expression, giving:

$$\begin{aligned}
\sigma_x &= \frac{E}{1-3v^2-2v^3}\ \big((1-v^2)\varepsilon_x + (v+v^2)\varepsilon_y + (v+v^2)\varepsilon_z\big) \\
\sigma_y &= \frac{E}{1-3v^2-2v^3}\ \big((v+v^2)\varepsilon_x + (1-v^2)\varepsilon_y + (v+v^2)\varepsilon_z\big) \\
\sigma_z &= \frac{E}{1-3v^2-2v^3}\ \big((v+v^2)\varepsilon_x + (v+v^2)\varepsilon_y + (1-v^2)\varepsilon_z\big)
\end{aligned}$$

Interpretation

Matrix manipulation has been used to transform three simultaneous equations relating strain to stress into simultaneous equations relating stress to strain in terms of the elastic constants. These would be useful for deducing the applied stress if the strains are known. The original equations enable calculation of strains if the applied stresses are known.

Exercises

1. Solve the following using Cramer's rule:

(a) $\begin{aligned} 2x - 3y &= 1 \\ 4x + 4y &= 2 \end{aligned}$ (b) $\begin{aligned} 2x - 5y &= 2 \\ -4x + 10y &= 1 \end{aligned}$ (c) $\begin{aligned} 6x - y &= 0 \\ 2x - 4y &= 1 \end{aligned}$

2. Using Cramer's rule obtain the solutions to the following sets of equations:

(a) $\begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ x_1 \qquad + x_3 &= 4 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$ (b) $\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 \qquad + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$

Answers

1. (a) $x = \dfrac{1}{2},\ y = 0$ (b) $\Delta = 0$, no solution (c) $x = -\dfrac{1}{22},\ y = -\dfrac{3}{11}$

2. (a) $x_1 = \dfrac{8}{3},\ x_2 = -4,\ x_3 = \dfrac{4}{3}$ (b) $x_1 = \dfrac{1}{2},\ x_2 = 1,\ x_3 = \dfrac{3}{2}$