Matrix Multiplication 7.2

Introduction

When we wish to multiply matrices together we have to ensure that the operation is possible - and this is not always so. Also, unlike number arithmetic and algebra, even when the product exists the order of multiplication may have an effect on the result. In this Section we pick our way through the minefield of matrix multiplication.

Prerequisites

Before starting this Section you should ...

- understand the concept of a matrix and associated terms.

Learning Outcomes

On completion you should be able to ...

- decide when the product AB exists
- recognise that $AB \neq BA$ in most cases
- carry out the multiplication AB
- explain what is meant by the identity matrix I

1. Multiplying row matrices and column matrices together

Let A be a 1×2 row matrix and B be a 2×1 column matrix:

$$A = \begin{bmatrix} a & b \end{bmatrix} \qquad B = \begin{bmatrix} c \\ d \end{bmatrix}$$

The product of these two matrices is written AB and is the 1×1 matrix defined by:

$$AB = \begin{bmatrix} a & b \end{bmatrix} \times \begin{bmatrix} c \\ d \end{bmatrix} = [ac + bd]$$

Note that corresponding elements are multiplied together and the results are then added together. For example

$$\begin{bmatrix} 2 & -3 \end{bmatrix} \times \begin{bmatrix} 6 \\ 5 \end{bmatrix} = [12 - 15] = [-3]$$

This matrix product is easily generalised to other row and column matrices. For example if C is a 1×4 row matrix and D is a 4×1 column matrix:

$$C = \begin{bmatrix} 2 & -4 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 5 \end{bmatrix}$$

then we define the product of C with D as

$$CD = \begin{bmatrix} 2 & -4 & 3 & 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 3 \\ -2 \\ 5 \end{bmatrix} = [6 - 12 - 6 + 10] = [-2]$$

The only requirement is that the number of elements of the row matrix is the same as the number of elements of the column matrix.

2. Multiplying two 2×2 matrices

If A and B are two matrices then the product AB is obtained by multiplying the rows of A with the columns of B in the manner described above. This will only be possible if the number of elements in the rows of A is the same as the number of elements in the columns of B. In particular, we define the product of two 2×2 matrices A and B to be another 2×2 matrix C whose elements are calculated according to the following pattern

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} w & x \\ y & z \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}}_{C}$$

The rule for calculating the elements of C is described in the following Key Point:

Key Point 4

Matrix Product

$$AB = C$$

The element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B.

We illustrate this construction for the abstract matrices A and B given above:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} [\,a \;\; b\,]\begin{bmatrix} w \\ y \end{bmatrix} & [\,a \;\; b\,]\begin{bmatrix} x \\ z \end{bmatrix} \\ \\ [\,c \;\; d\,]\begin{bmatrix} w \\ y \end{bmatrix} & [\,c \;\; d\,]\begin{bmatrix} x \\ z \end{bmatrix} \end{bmatrix} = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}$$

For example

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} [\,2 \;\; -1\,]\begin{bmatrix} 2 \\ 6 \end{bmatrix} & [\,2 \;\; -1\,]\begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ \\ [\,3 \;\; -2\,]\begin{bmatrix} 2 \\ 6 \end{bmatrix} & [\,3 \;\; -2\,]\begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ -6 & 10 \end{bmatrix}$$

Task

Find the product AB where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$

First write down row 1 of A, column 2 of B and form the first element in product AB:

Your solution

Answer

$[1, 2]$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$; their product is $1 \times (-1) + 2 \times 1 = 1$.

Now repeat the process for row 2 of A, column 1 of B:

Your solution

Answer

$[3, 4]$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Their product is $3 \times 1 + 4 \times (-2) = -5$

Finally find the two other elements of $C = AB$ and hence write down the matrix C:

Your solution

Answer

Row 1 column 1 is $1 \times 1 + 2 \times (-2) = -3$. Row 2 column 2 is $3 \times (-1) + 4 \times 1 = 1$

$C = \begin{bmatrix} -3 & 1 \\ -5 & 1 \end{bmatrix}$

Clearly, matrix multiplication is tricky and not at all 'natural'. However, it is a very important mathematical procedure with many engineering applications so must be mastered.

3. Some surprising results

We have already calculated the product AB where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

Now complete the following task in which you are asked to determine the product BA, i.e. with the matrices in reverse order.

Task

For matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ form the products of

row 1 of B and column 1 of A row 1 of B and column 2 of A

row 2 of B and column 1 of A row 2 of B and column 2 of A

Now write down the matrix BA:

Your solution

Answer

row 1, column 1 is $1 \times 1 + (-1) \times 3 = -2$ row 1, column 2 is $1 \times 2 + (-1) \times 4 = -2$

row 2, column 1 is $-2 \times 1 + 1 \times 3 = 1$ row 2, column 2 is $-2 \times 2 + 1 \times 4 = 0$

BA is $\begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$

It is clear that AB and BA are **not** in general the same. In fact it is the **exception** that $AB = BA$. In the special case in which $AB = BA$ we say that the matrices A and B **commute**.

Task

Calculate AB and BA where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Your solution

Answer

$$AB = BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We call B the 2×2 **zero matrix** written $\underline{0}$ so that $A \times \underline{0} = \underline{0} \times A = 0$ for any matrix A.

Now in the multiplication of numbers, the equation

$$ab = 0$$

implies that either a is zero or b is zero or both are zero. The following task shows that this is not necessarily true for matrices.

Task

Carry out the multiplication AB where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Your solution

Answer

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here we have a zero product yet neither A nor B is the zero matrix! Thus the statement $AB = 0$ does **not** allow us to conclude that either $A = \underline{0}$ or $B = \underline{0}$.

Task

Find the product AB where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Your solution

Answer

$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the **identity matrix** or **unit matrix** of order 2, and is usually denoted by the symbol I. (Strictly we should write I_2, to indicate the size.) I plays the same role in matrix multiplication as the number 1 does in number multiplication.

Hence

just as $a \times 1 = 1 \times a = a$ for any number a, so $AI = IA = A$ for any matrix A.

4. Multiplying two 3×3 matrices

The definition of the product $C = AB$ where A and B are two 3×3 matrices is as follows

$$C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} ar+bu+cx & as+bv+cy & at+bw+cz \\ dr+eu+fx & ds+ev+fy & dt+ew+fz \\ gr+hu+ix & gs+hv+iy & gt+hw+iz \end{bmatrix}$$

This looks a rather daunting amount of algebra but in fact the construction of the matrix on the right-hand side is straightforward if we follow the simple rule from Key Point 4 that the element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B.

For example, to obtain the element in row 2, column 3 of C we take row 2 of A: $[d, e, f]$ and multiply it with column 3 of B in the usual way to produce $[dt + ew + fz]$.

By repeating this process we obtain every element of C.

Task

Calculate $AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

First find the element in row 2 column 1 of the product:

Your solution

Answer

Row 2 of A is $(3, 4, 0)$ column 1 of B is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

The combination required is $3 \times 2 + 4 \times 1 + (0) \times (0) = 10$.

Now complete the multiplication to find all the elements of the matrix AB:

Your solution

Answer

In full detail, the elements of AB are:

$$\begin{bmatrix} 1 \times 2 + 2 \times 1 + (-1) \times 0 & 1 \times (-1) + 2 \times (-2) + (-1) \times 3 & 1 \times 3 + 2 \times 1 + (-1) \times (-2) \\ 3 \times 2 + 4 \times 1 + 0 \times 0 & 3 \times (-1) + 4 \times (-2) + 0 \times 3 & 3 \times 3 + 4 \times 1 + 0 \times (-2) \\ 1 \times 2 + 5 \times 1 + (-2) \times 0 & 1 \times (-1) + 5 \times (-2) + (-2) \times 3 & 1 \times 3 + 5 \times 1 + (-2) \times (-2) \end{bmatrix}$$

i.e. $AB = \begin{bmatrix} 4 & -8 & 7 \\ 10 & -11 & 13 \\ 7 & -17 & 12 \end{bmatrix}$

The 3×3 unit matrix is: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and as in the 2×2 case this has the property that $AI = IA = A$

The 3×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. Multiplying non-square matrices together

So far, we have just looked at multiplying 2×2 matrices and 3×3 matrices. However, products between non-square matrices may be possible.

Key Point 5

General Matrix Products

The general rule is that an $n \times p$ matrix A can be multiplied by a $p \times m$ matrix B to form an $n \times m$ matrix $AB = C$.

In words:

For the matrix product AB to be defined the number of columns of A must equal the number of rows of B.

The elements of C are found in the usual way:

The element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B.

Example 4

Find the product AB if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 6 & 1 \\ 4 & 3 \end{bmatrix}$

Solution

Since A is a 2×3 and B is a 3×2 matrix the product AB can be found and results in a 2×2 matrix.

$$AB = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 6 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 22 & 13 \\ 38 & 25 \end{bmatrix}$$

Task

Obtain the product AB if $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 1 \\ 6 & 1 & 0 \end{bmatrix}$

Your solution

Answer

AB is a 2×3 matrix.

$$AB = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 1 \\ 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \\ \begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 1 \\ -14 & 5 & 2 \end{bmatrix}$$

6. The rules of matrix multiplication

It is worth noting that the process of multiplication can be continued to form products of more than two matrices.

Although two matrices may not commute (i.e. in general $AB \neq BA$) the **associative law** always holds i.e. for matrices which **can** be multiplied,

$$A(BC) = (AB)C.$$

The general principle is **keep the left to right order**, but within that limitation any two adjacent matrices can be multiplied.

It is important to note that it is not always possible to multiply together any two given matrices.

For example if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ then $AB = \begin{bmatrix} a+2d & b+2e & c+2f \\ 3a+4d & 3b+4e & 3c+4f \end{bmatrix}$.

However $BA = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is **not defined** since each row of B has three elements whereas each column of A has two elements and we cannot multiply these elements in the manner described.

Task

Given $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

State which of the products $AB,\ BA,\ AC,\ CA,\ BC,\ CB,\ (AB)C,\ A(CB)$ is defined and state the size $(n \times m)$ of the product when defined.

Your solution

AB
BA
AC
CA
BC
CB
$(AB)C$
$A(CB)$

Answer

$\begin{array}{cc} A & B \\ 2\times 3 & 2\times 2 \end{array}$	not possible	$\begin{array}{cc} B & A \\ 2\times 2 & 2\times 3 \end{array}$	possible; result 2×3
$\begin{array}{cc} A & C \\ 2\times 3 & 3\times 2 \end{array}$	possible; result 2×2	$\begin{array}{cc} C & A \\ 3\times 2 & 2\times 3 \end{array}$	possible; result 3×3
$\begin{array}{cc} B & C \\ 2\times 2 & 3\times 2 \end{array}$	not possible	$\begin{array}{cc} C & B \\ 3\times 2 & 2\times 2 \end{array}$	possible; result 3×2
$(AB)C$	not possible, AB not defined.	$\begin{array}{cc} A & (C\ B) \\ 2\times 3 & 3\times 2 \end{array}$	possible; result 2×2

We now list together some properties of matrix multiplication and compare them with corresponding properties for multiplication of numbers.

Key Point 6

Matrix algebra	Number algebra
$A(B+C) = AB + AC$	$a(b+c) = ab + ac$
$AB \neq BA$ in general	$ab = ba$
$A(BC) = (AB)C$	$a(bc) = (ab)c$
$AI = IA = A$	$1.a = a.1 = a$
$A\underline{0} = \underline{0}A = \underline{0}$	$0.a = a.0 = 0$
AB may not be possible	ab is always possible
$AB = \underline{0}$ does not imply $A = \underline{0}$ or $B = \underline{0}$	$ab = 0 \ \rightarrow \ a = 0$ or $b = 0$

Application of matrices to networks

A network is a collection of points (nodes) some of which are connected together by lines (paths). The information contained in a network can be conveniently stored in the form of a matrix.

Example 5

Petrol is delivered to terminals T_1 and T_2. They distribute the fuel to 3 storage depots ($S_1,\ S_2,\ S_3$). The network diagram below shows what fraction of the fuel goes from each terminal to the three storage depots. In turn the 3 depots supply fuel to 4 petrol stations (P_1, P_2, P_3, P_4) as shown in Figure 2:

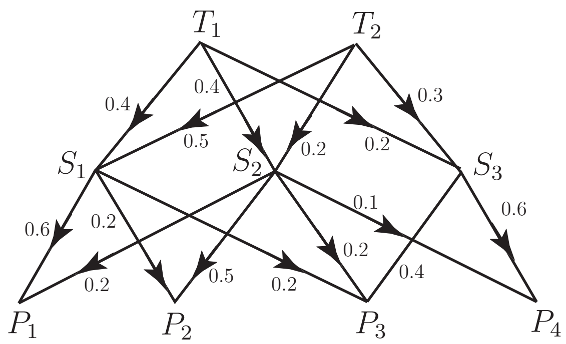
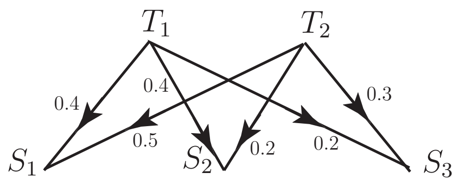


Figure 2

Show how this situation may be described using matrices.

Solution

Denote the amount of fuel, in litres, flowing from T_1 by t_1 and from T_2 by t_2 and the quantity being received at S_i by s_i for $i = 1, 2, 3$. This situation is described in the following diagram:

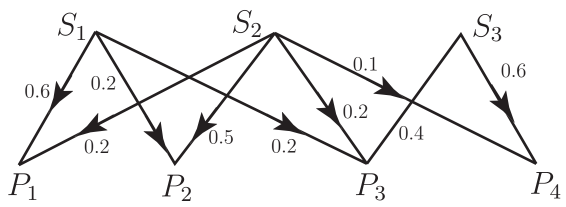


From this diagram we see that

$$\begin{aligned} s_1 &= 0.4t_1 + 0.5t_2 \\ s_2 &= 0.4t_1 + 0.2t_2 \\ s_3 &= 0.2t_1 + 0.3t_2 \end{aligned} \qquad \text{or, in matrix form:} \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.2 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

Solution (contd.)

In turn the 3 depots supply fuel to 4 petrol stations as shown in the next diagram:



If the petrol stations receive p_1, p_2, p_3, p_4 litres respectively then from the diagram we have:

$$\begin{aligned} p_1 &= 0.6s_1 + 0.2s_2 \\ p_2 &= 0.2s_1 + 0.5s_2 \\ p_3 &= 0.2s_1 + 0.2s_2 + 0.4s_3 \\ p_4 &= 0.1s_2 + 0.6s_3 \end{aligned} \quad \text{or, in matrix form:} \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.5 & 0 \\ 0.2 & 0.2 & 0.4 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Combining the equations, substituting expressions for s_1, s_2, s_3 in the equations for p_1, p_2, p_3, p_4

we get:

$$\begin{aligned} p_1 &= 0.6s_1 + 0.2s_2 \\ &= 0.6(0.4t_1 + 0.5t_2) + 0.2(0.4t_1 + 0.2t_1) \\ &= 0.32t_1 + 0.34t_2 \end{aligned}$$

with similar results for p_2, p_3 and p_4.

This is equivalent to combining the two networks. The results can be obtained more easily by multiplying the matrices:

$$\begin{aligned} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} &= \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.5 & 0 \\ 0.2 & 0.2 & 0.4 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\ &= \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.5 & 0 \\ 0.2 & 0.2 & 0.4 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.2 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \\ &= \begin{bmatrix} 0.32 & 0.34 \\ 0.28 & 0.20 \\ 0.24 & 0.26 \\ 0.16 & 0.20 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0.32t_1 + 0.34t_2 \\ 0.28t_1 + 0.20t_2 \\ 0.24t_1 + 0.26t_2 \\ 0.16t_1 + 0.20t_2 \end{bmatrix} \end{aligned}$$

Engineering Example 1

Communication network

Problem in words

Figure 3 represents a communication network. Vertices a, b, f and g represent offices. Vertices c, d and e represent switching centres. The numbers marked along the edges represent the number of connections between any two vertices. Calculate the number of routes from a and b to f and g

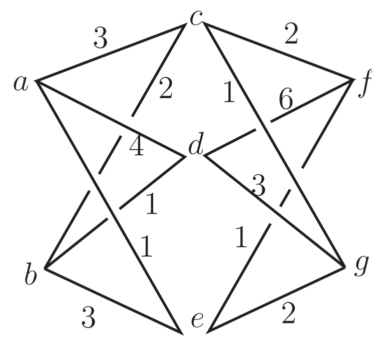


Figure 3: A communication network where a, b, f and g are offices and c, d and e are switching centres

Mathematical statement of the problem

The number of routes from a to f can be calculated by taking the number via c plus the number via d plus the number via e. In each case this is given by multiplying the number of connections along the edges connecting a to c, c to f etc. This gives the result:
Number of routes from a to $f = 3 \times 2 + 4 \times 6 + 1 \times 1 = 31$.

The nature of matrix multiplication means that the number of routes is obtained by multiplying the matrix representing the number of connections from ab to cde by the matrix representing the number of connections from cde to fg.

Mathematical analysis

The matrix representing the number of routes from ab to cde is:

$$\begin{array}{cc} & \begin{array}{ccc} c & d & e \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 3 \end{pmatrix} \end{array}$$

The matrix representing the number of routes from cde to fg is:

$$\begin{array}{cc} & \begin{array}{cc} f & g \end{array} \\ \begin{array}{c} c \\ d \\ e \end{array} & \begin{pmatrix} 2 & 1 \\ 6 & 3 \\ 1 & 2 \end{pmatrix} \end{array}$$

The product of these two matrices gives the total number of routes.

$$\begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 6 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3\times 2+4\times 6+1\times 1 & 3\times 1+4\times 3+1\times 2 \\ 2\times 2+1\times 6+3\times 1 & 2\times 1+1\times 3+3\times 2 \end{pmatrix} = \begin{pmatrix} 31 & 17 \\ 13 & 11 \end{pmatrix}$$

Interpretation
We can interpret the resulting (product) matrix by labelling the columns and rows.

$$\begin{array}{c} \\ a \\ b \end{array}\begin{array}{c} \begin{array}{cc} f & g \end{array} \\ \begin{pmatrix} 31 & 17 \\ 13 & 11 \end{pmatrix} \end{array}$$

Hence there are 31 routes from a to f, 17 from a to g, 13 from b to f and 11 from b to g.

Exercises

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ $\quad C = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$ find

   (a) AB, (b) AC, (c) $(A+B)C$, (d) $AC+BC$ (e) $2A-3C$

2. If a rotation through an angle θ is represented by the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and a second rotation through an angle ϕ is represented by the matrix $B = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ show that both AB and BA represent a rotation through an angle $\theta+\phi$.

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ -1 & 2 \\ 5 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find AB and BC.

4. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, $\quad C = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$,

   verify $A(BC) = (AB)C$.

5. If $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$ then show that AA^T is symmetric.

6. If $A = \begin{bmatrix} 11 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ verify that $(AB)^T = \begin{bmatrix} 0 & 1 \\ 11 & 3 \\ 22 & 7 \end{bmatrix} = B^TA^T$

Answers

1. (a) $AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$ (b) $AC = \begin{bmatrix} 4 & -7 \\ 8 & -15 \end{bmatrix}$ (c) $(A+B)C = \begin{bmatrix} 16 & -30 \\ 24 & -46 \end{bmatrix}$

   (d) $AC+BC = \begin{bmatrix} 16 & -30 \\ 24 & -46 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 7 \\ 0 & 17 \end{bmatrix}$

2. $$AB = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}$$

   $$= \begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$$

   which clearly represents a rotation through angle $\theta+\phi$. BA gives the same result.

3. $AB = \begin{bmatrix} 15 & 26 \\ -6 & -12 \\ 12 & 24 \end{bmatrix}$, $\quad BC = \begin{bmatrix} 8 & 10 \\ 0 & 3 \\ 16 & 17 \end{bmatrix}$

4. $A(BC) = (AB)C = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$