Log-linear Graphs

6.6

Introduction

In this Section we employ our knowledge of logarithms to simplify plotting the relation between one variable and another. In particular we consider those situations in which one of the variables requires scaling because the range of its data values is very large in comparison to the range of the other variable.

We will only employ logarithms to base 10. To aid the plotting process we explain how log-linear graph paper is used. Unlike ordinary graph paper, one of the axes is scaled using logarithmic values instead of the values themselves. By this process, values which range from (say) 1 to 1,000,000 are scaled down to range over the values 0 to 6. We do not discuss log-log graphs, in which both data sets require scaling, as the reader will easily be able to adapt the technique described here to those situations.

Prerequisites

Before starting this Section you should . . .

- be familiar with the laws of logarithms
- have knowledge of logarithms to base 10
- be able to solve equations involving logarithms

Learning Outcomes

On completion you should be able to . . .

- decide when to use log-linear graph paper
- use log-linear graph paper to analyse functions of the form $y = ka^{px}$

1. Logarithms and scaling

In this Section we shall work entirely with logarithms to base 10.

We are already familiar with a particular property of logarithms: $\log A^k = k \log A$.

Now, choosing $A = 10$ we see that: $\log 10^k = k \log 10 = k$.

The effect of taking a logarithm is to replace a power: 10^k (which could be very large) by the value of the exponent k. Thus a range of numbers extending from 1 to 1,000,000 say, can be transformed, by taking logarithms to base 10, into a range of numbers from 0 to 6. This approach is especially useful in the exercise of plotting one variable against another in which one of the variables has a wide range of values.

Example 10

Plot the following values (x, y)

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	1.0	2.14	4.3	8.16	14.8	25.6	42.9

Estimate the value of y when $x = 1.35$.

Solution

If we attempt to plot these values on ordinary graph paper in which both vertical and horizontal scales are linear we find the large range in the y-values presents a problem. The values near the lower end are bunched together and interpolating to find the value of y when $x = 1.35$ is difficult.

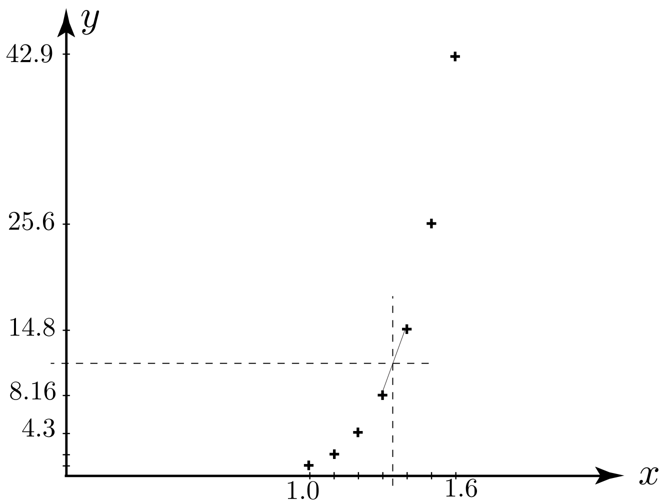


Figure 12

Example 11

To alleviate the scaling problem in Example 10 employ logarithms to scale down the y-values, giving:

x	1	1.1	1.2	1.3	1.4	1.5	1.6
$\log y$	0	0.33	0.63	0.97	1.17	1.41	1.63

Plot these values and estimate the value of y when $x = 1.35$.

Solution

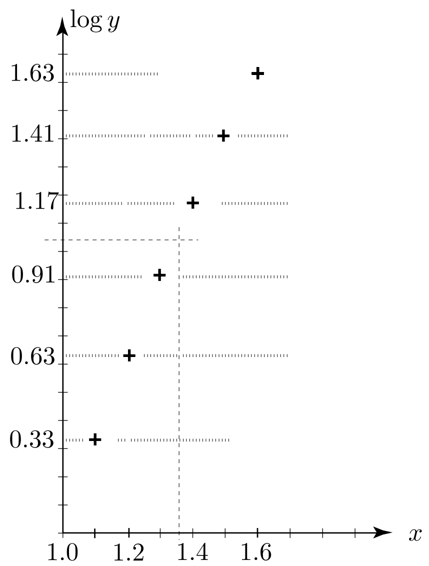


Figure 13

This approach has spaced-out the vertical values allowing a much easier assessment for the value of y at $x = 1.35$. From the graph we see that at $x = 1.35$ the '$\log y$' value is approximately 1.05. Taking $\log y = 1.05$ and inverting we get

$$y = 10^{1.05} = 11.22$$

2. Log-linear graph paper

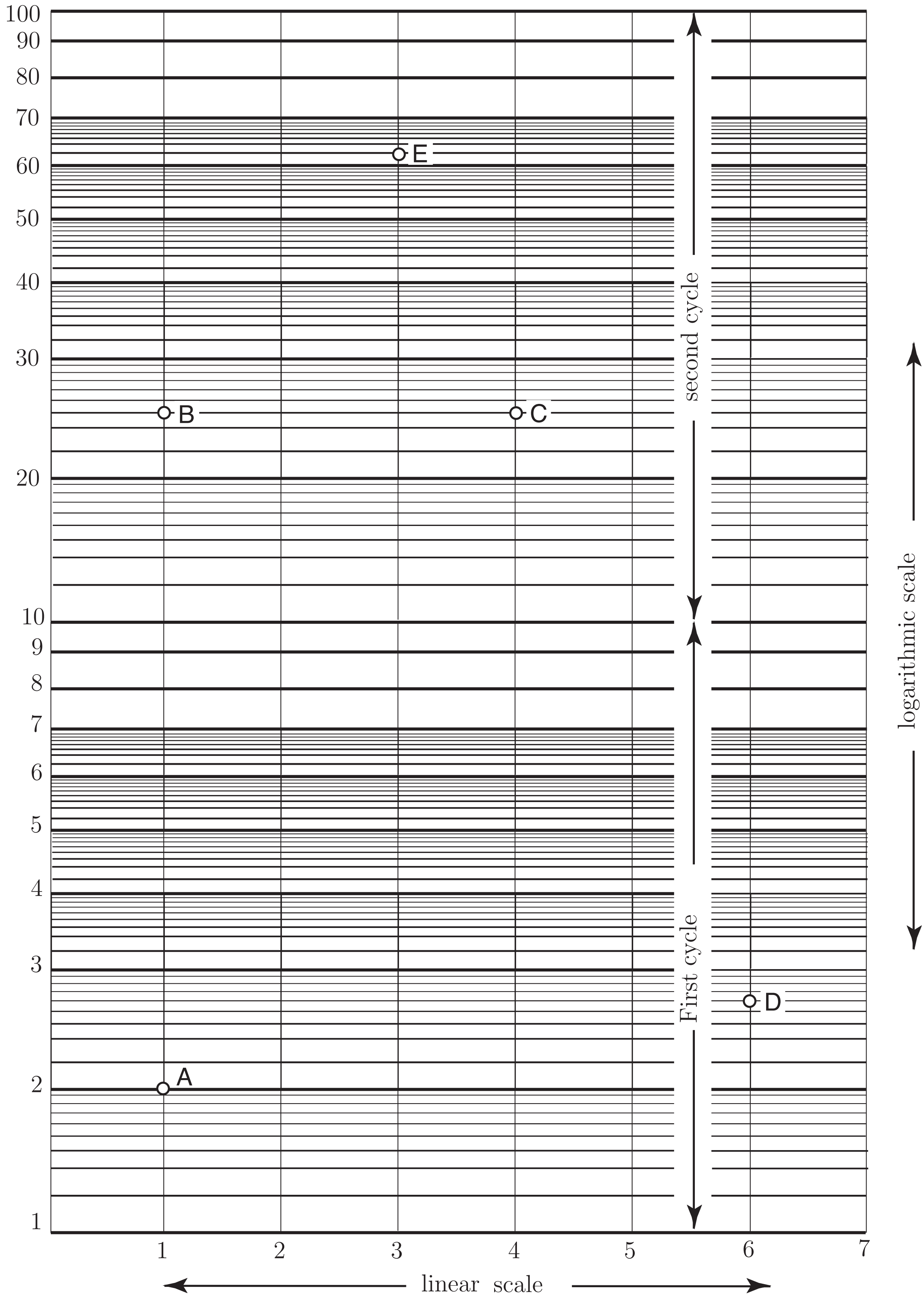
Ordinary graph paper has **linear** scales in both the horizontal (x) and vertical (y) directions. As we have seen, this can pose problems if the range of one of the variables, y say, is very large. One way round this is to take the **logarithm** of the y-values and re-plot on ordinary graph paper. Another common approach is to use **log-linear graph paper** in which the vertical scale is a **non-linear logarithmic scale**. Use of this special graph paper means that the original data can be plotted directly without the need to convert to logarithms which saves time and effort.

In log-linear graph paper the vertical axis is divided into a number of **cycles**. Each cycle corresponds to a jump in the data values by a factor of 10. For example, if the range of y-values extends from (say) 1 to 100 (or equivalently 10^0 to 10^2) then 2-cycle log-linear paper would be required. If the y-values extends from (say) 100 to 100,000 (or equivalently from 10^2 to 10^5) then 3-cycle log-linear paper would be used. Some other examples are given in Table 3:

Table 3

$y -$ values	$\log y$ values	no. of cycles
$1 \rightarrow 10$	$0 \rightarrow 1$	1
$1 \rightarrow 100$	$0 \rightarrow 2$	2
$10 \rightarrow 10,000$	$1 \rightarrow 4$	3
$\frac{1}{10} \rightarrow 100$	$-1 \rightarrow 2$	3

An example of 2-cycle log-linear graph paper is shown in Figure 14. We see that the horizontal scale is linear. The vertical scale is divided by lines denoted by 1,2,3,. . . ,10,20,30,. . . ,100. In the first cycle each of the horizontal blocks (separated by a slightly thicker line) is also divided according to a log-linear scale; so, for example, in the range $1 \rightarrow 2$ we have 9 horizontal lines representing the values 1.1, 1.2, . . . , 1.9. These subdivisions have been repeated (appropriately scaled) in blocks 2-3, 3-4, 4-5, 5-6, 6-7. The subdivisions have been omitted from blocks 7-8, 8-9, 9-10 for reasons of clarity. On this graph paper, we have noted the positions of $A:(1,2),\ B:(1,23),\ C:(4,23),\ D:(6,2.5),\ E:(3,61)$.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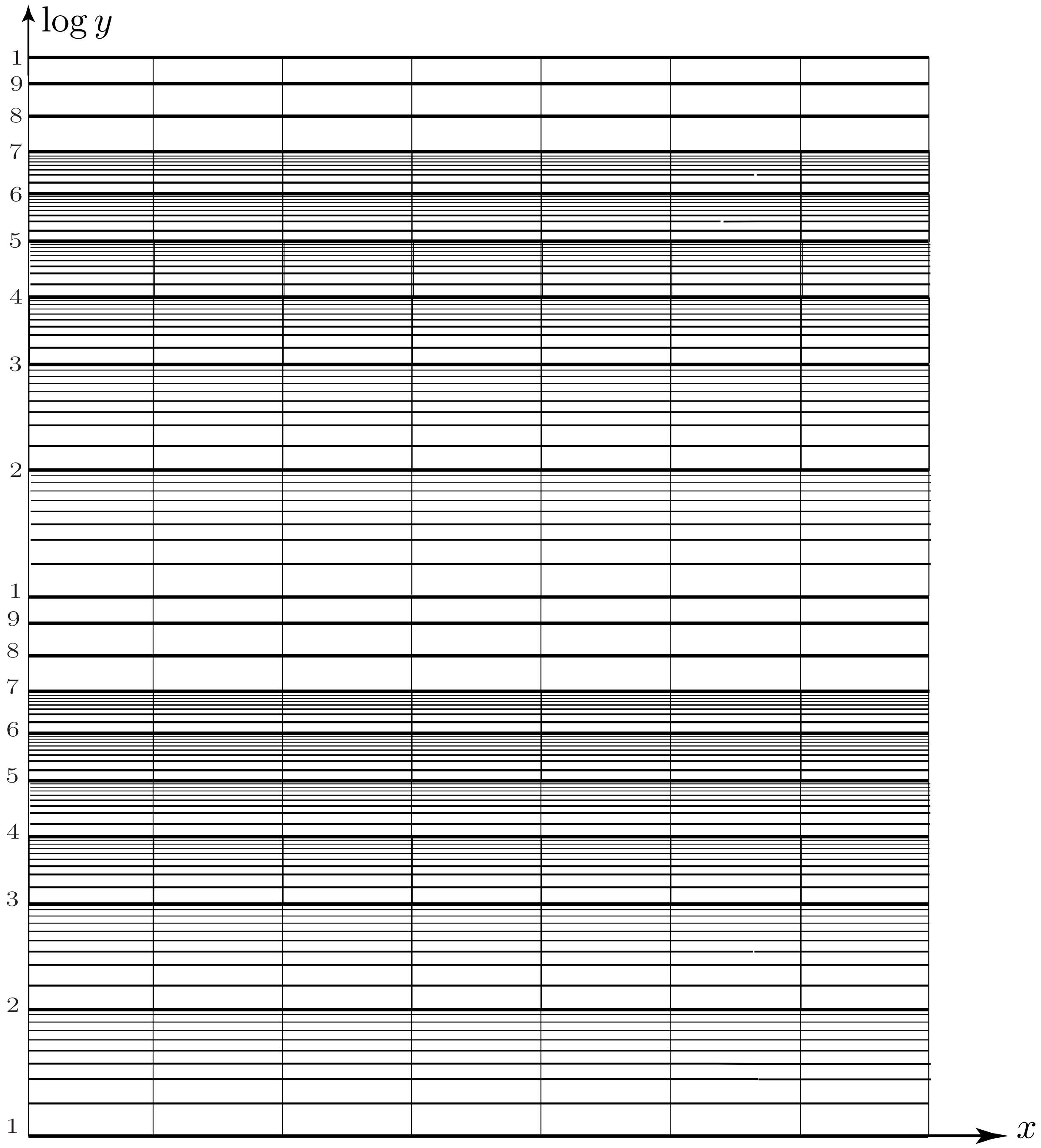
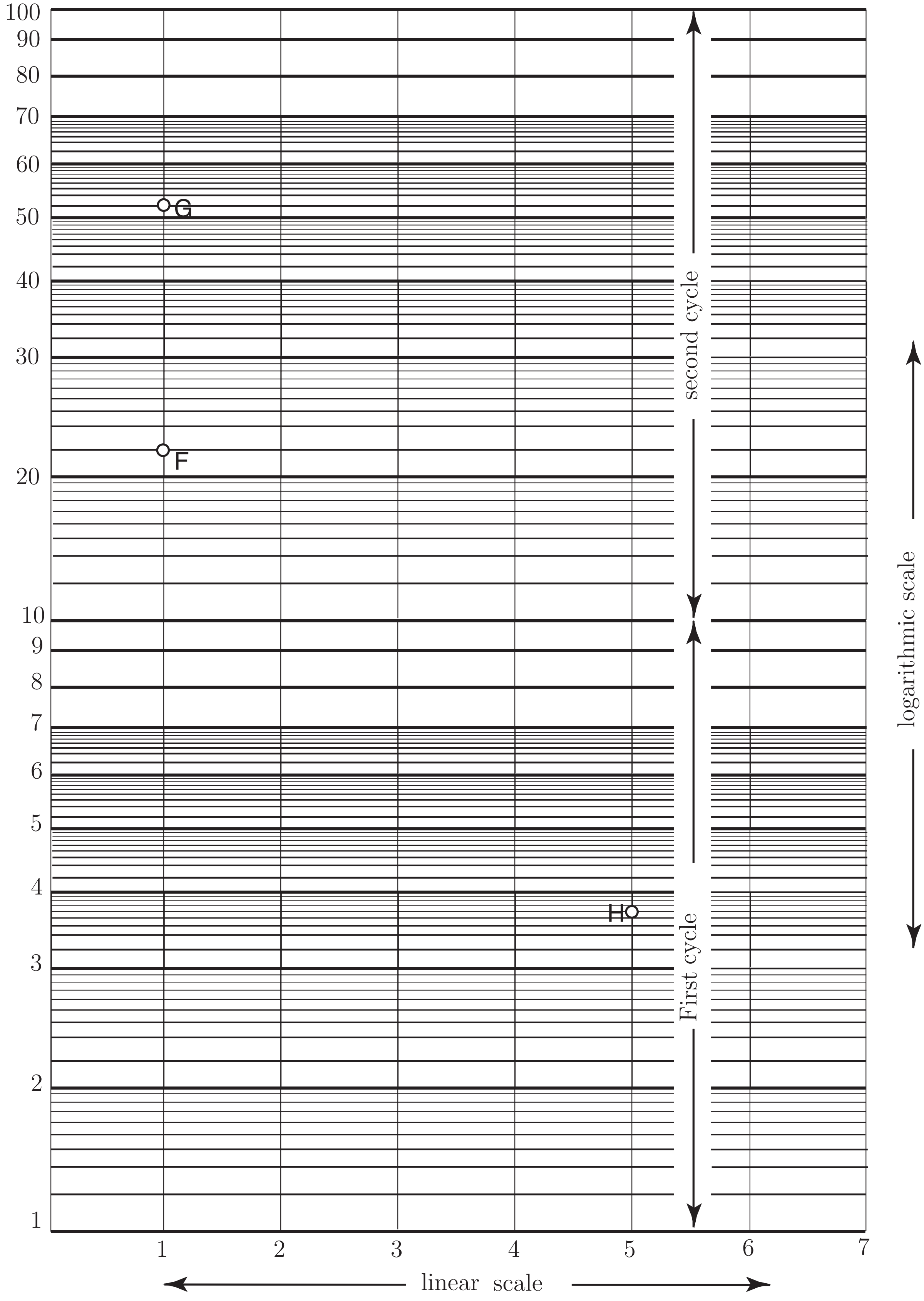
3
4
5
6
7
A
B
C
D
E
second cycle
First cycle
logarithmic scale
linear scale

Figure 14

Task

On the 2-cycle log-linear graph paper (below) locate the positions of the points $F:(2,21)$, $G:(2,51)$, $H:(5,3.5)$. [The correct positions are shown on the graph on next page.]

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1
G
F
H
second cycle
First cycle
logarithmic scale
1
2
3
4
5
6
7
linear scale

Example 12

It is thought that the relationship between two variables x, y is exponential

$$y = ka^x$$

An experiment is performed and the following pairs of data values (x, y) were obtained

x	1	2	3	4	5
y	5.9	12	26	49	96

Verify that the relation $y = ka^x$ is valid by plotting values on log-linear paper to obtain a set of points lying on a straight line. Estimate the values of k, a.

Solution

First we rearrange the relation $y = ka^x$ by taking logarithms (to base 10).

$$\therefore \qquad \log y = \log(ka^x) = \log k + x \log a$$

So, if we define a new variable $Y \equiv \log y$ then the relationship between Y and x will be linear – its graph (on log-linear paper) should be a straight line. The vertical intercept of this line is $\log k$ and the gradient of the line is $\log a$. Each of these can be obtained from the graph and the values of a, k inferred.

When using log-linear graphs, the reader should keep in mind that, on the vertical axis, the values are not as written but the logarithms of those values.

We have plotted the points and drawn a straight line (as best we can) through them - see Figure 15. (We will see in a later Workbook (HELM 31) how we might improve on this subjective approach to fitting straight lines to data points). The line intersects the vertical axis at a value $\log(3.13)$ and the gradient of the line is

$$\frac{\log 96 - \log 3.13}{5 - 0} = \frac{\log(96/3.13)}{5} = \frac{\log 30.67}{5} = 0.297$$

But the intercept is $\log k$ so

$$\log k = \log 3.13 \qquad \text{implying} \qquad k = 3.13$$

and the gradient is $\log a$ so

$$\log a = 0.297 \qquad \text{implying} \qquad a = 10^{0.297} = 1.98$$

We conclude that the relation between the x, y variables is well modelled by the relation $y = 3.13(1.98)^x$. If the points did not lie more-or-less on a straight line then we would conclude that the relationship was *not* of the form $y = ka^x$.

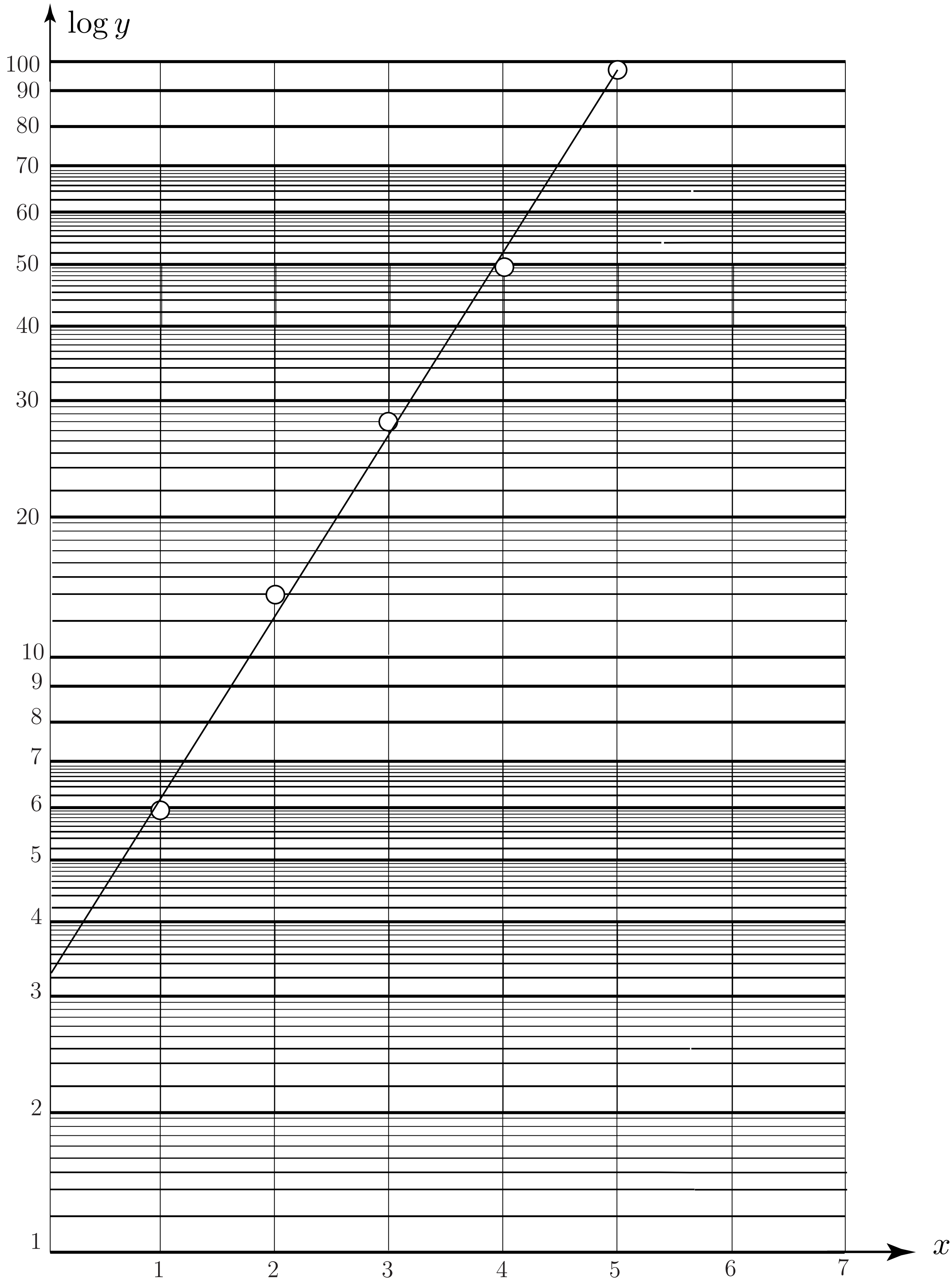


Figure 15

Task

Using a log-linear graph estimate the values of k, a if it is assumed that $y = ka^{-2x}$ and the data values connecting x, y are:

x	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3
y	190	155	123	100	80	63	52

First take logs of the relation $y = ka^{-2x}$ and introduce an appropriate new variable:

Your solution

$y = ka^{-2x}$ implies $\log y = \log(ka^{-2x}) =$

introduce $Y =$

$\log y = \log k - 2x \log a$. Let $Y = \log y$ then $Y = \log k + x(-2\log a)$. We therefore expect a linear relation between Y and x (i.e. on log-linear paper).

Now determine how many cycles are required in your log-linear paper:

Your solution

The range of values of y is 140; from 5.2×10 to 1.9×10^2. So 2-cycle log-linear paper is needed.

Now plot the data values directly onto log-linear paper (supplied on the next page) and decide whether the relation $y = ka^{-2x}$ is acceptable:

Your solution

It is acceptable. On plotting the points a straight line fits the data well which is what we expect from $Y = \log k + x(-2\log a)$.

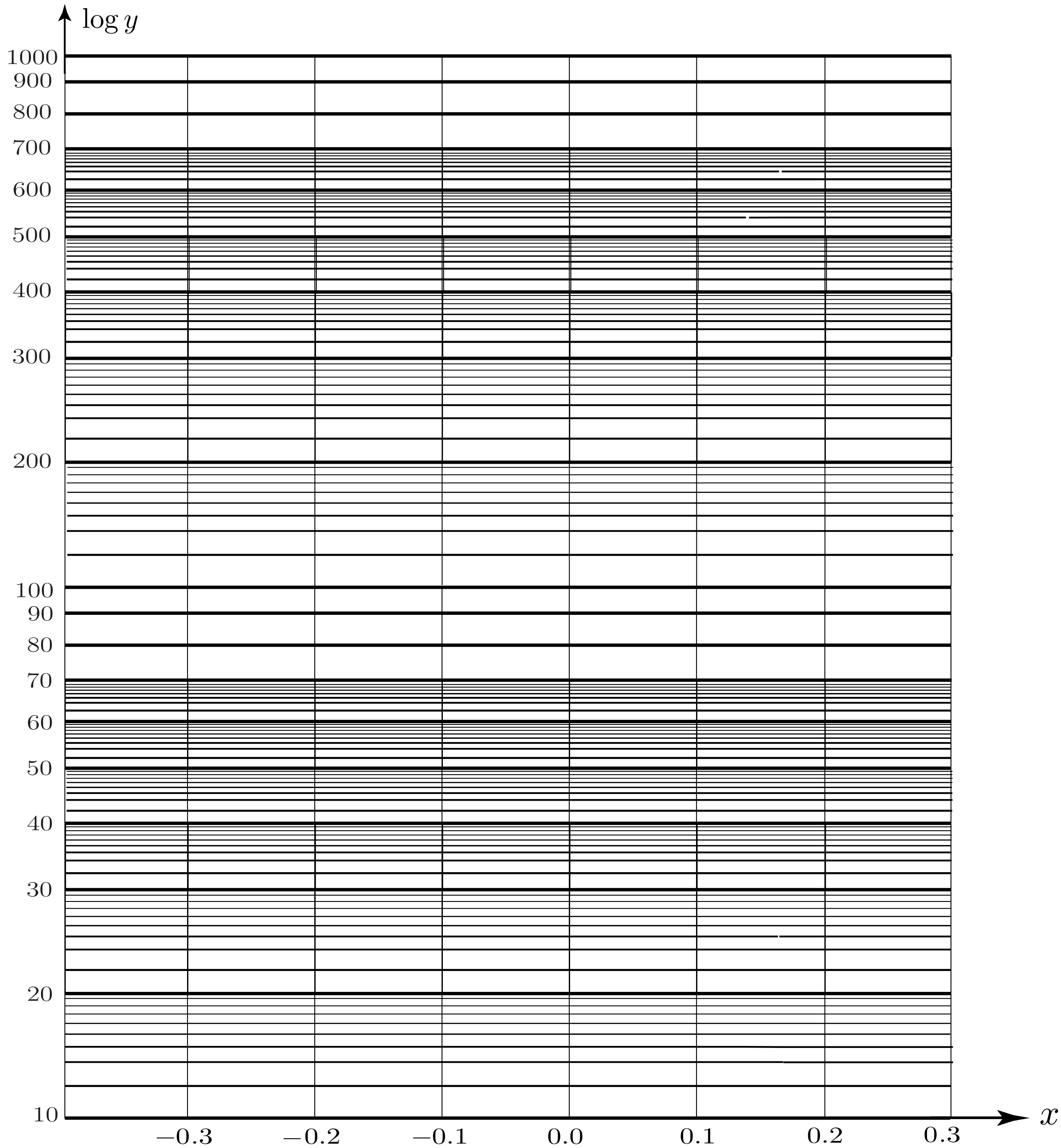
Now, using knowledge of the intercept and the gradient, find the values of k, a:

Your solution

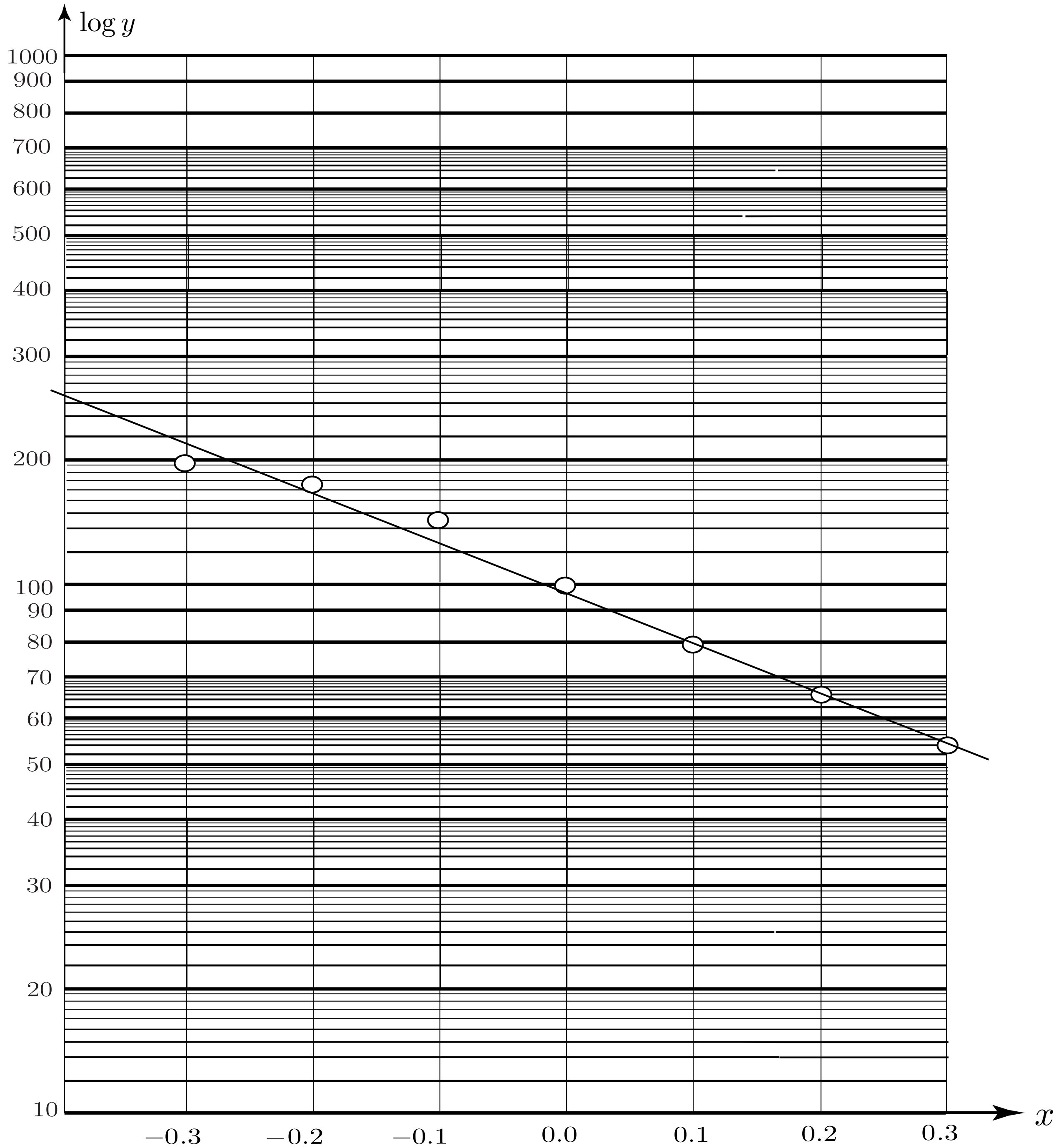
See the graph two pages further on. $k \approx 94$ (intercept on $x = 0$ line). The gradient is

$$\frac{\log 235 - \log 52}{-0.4 - 0.3} = -\frac{\log(235/52)}{0.7} = -\frac{0.655}{0.7} = -0.935$$

But the gradient is $-2\log a$. Thus $-2\log a = -0.935$ which implies $a = 10^{0.468} = 2.93$



Your solution to Task on page 67



Answer to Task on page 67

Use the log-linear graph sheets supplied on the following pages for these Exercises.

Exercises

1. Estimate the values of k and a if $y = ka^x$ represents the following set of data values:

x	0.5	1	2	3	4
y	5.93	8.8	19.36	42.59	93.70

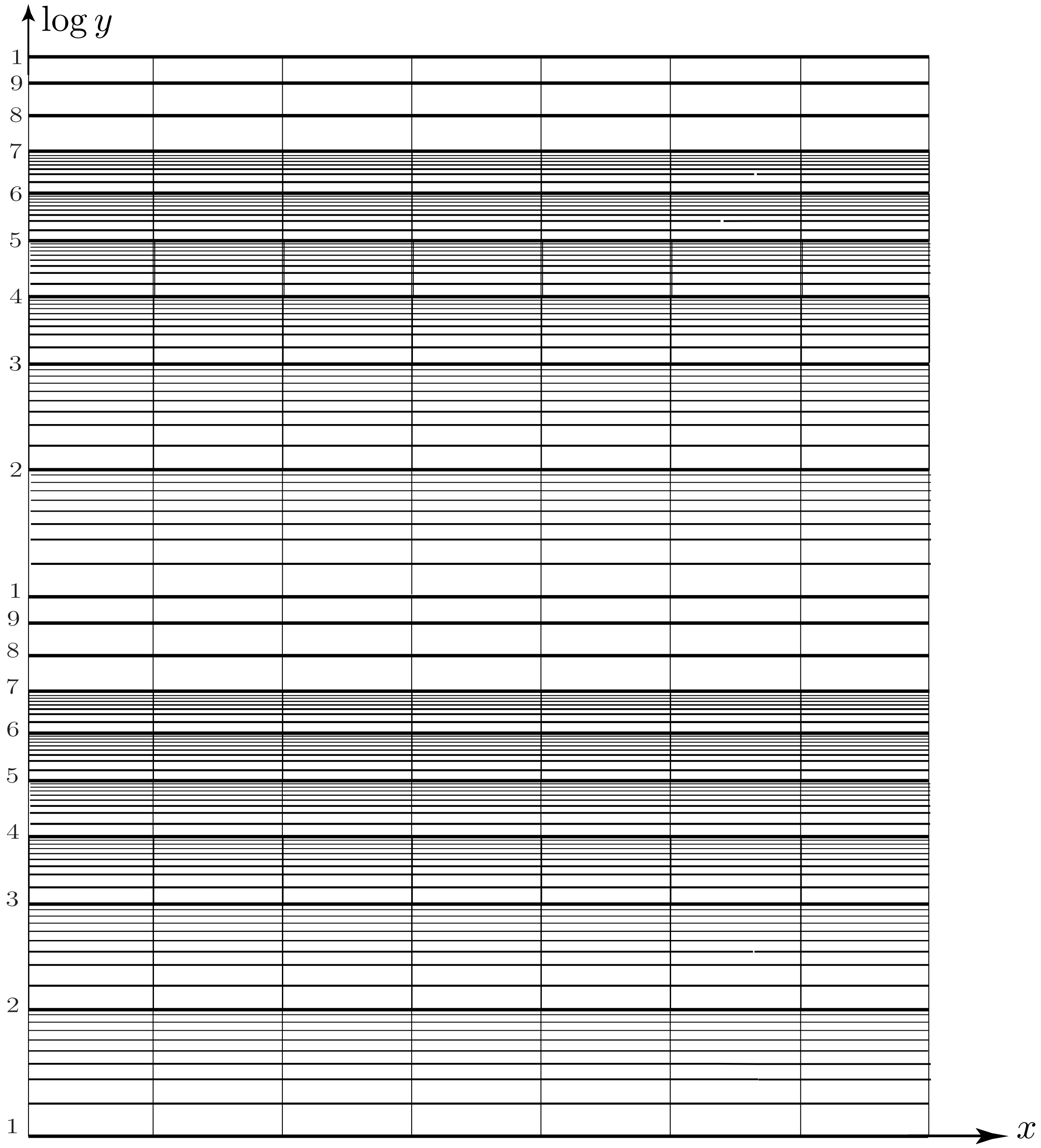
2. Estimate the values of k and a if the relation $y = k(a)^{-x}$ is a good representation for the data values:

x	2	2.5	3	3.5	4
y	7.9	3.6	1.6	0.7	0.3

Answers

1. $k \approx 4 \quad a \approx 2.2$

2. $k \approx 200 \quad a \approx 5$


log y
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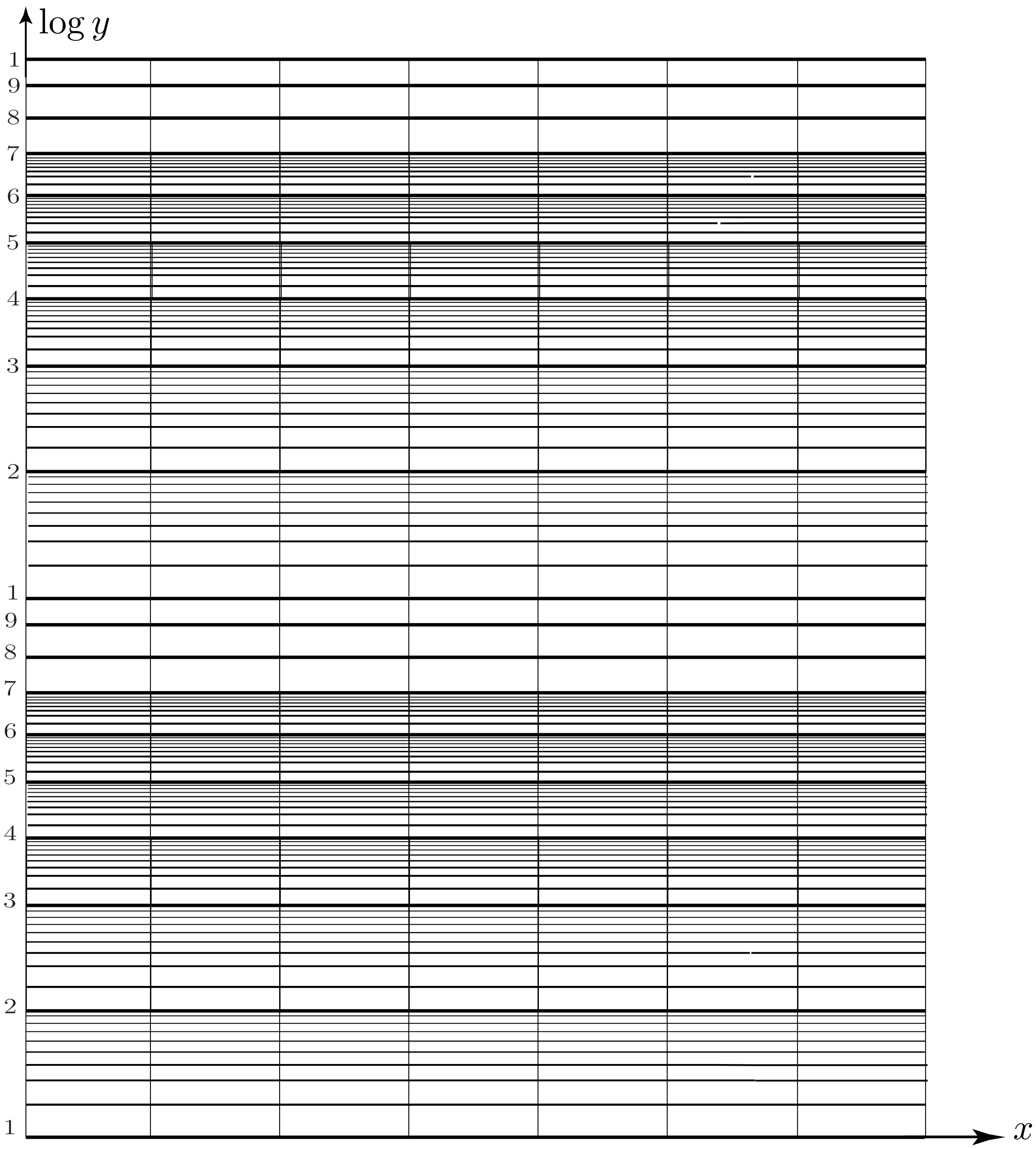
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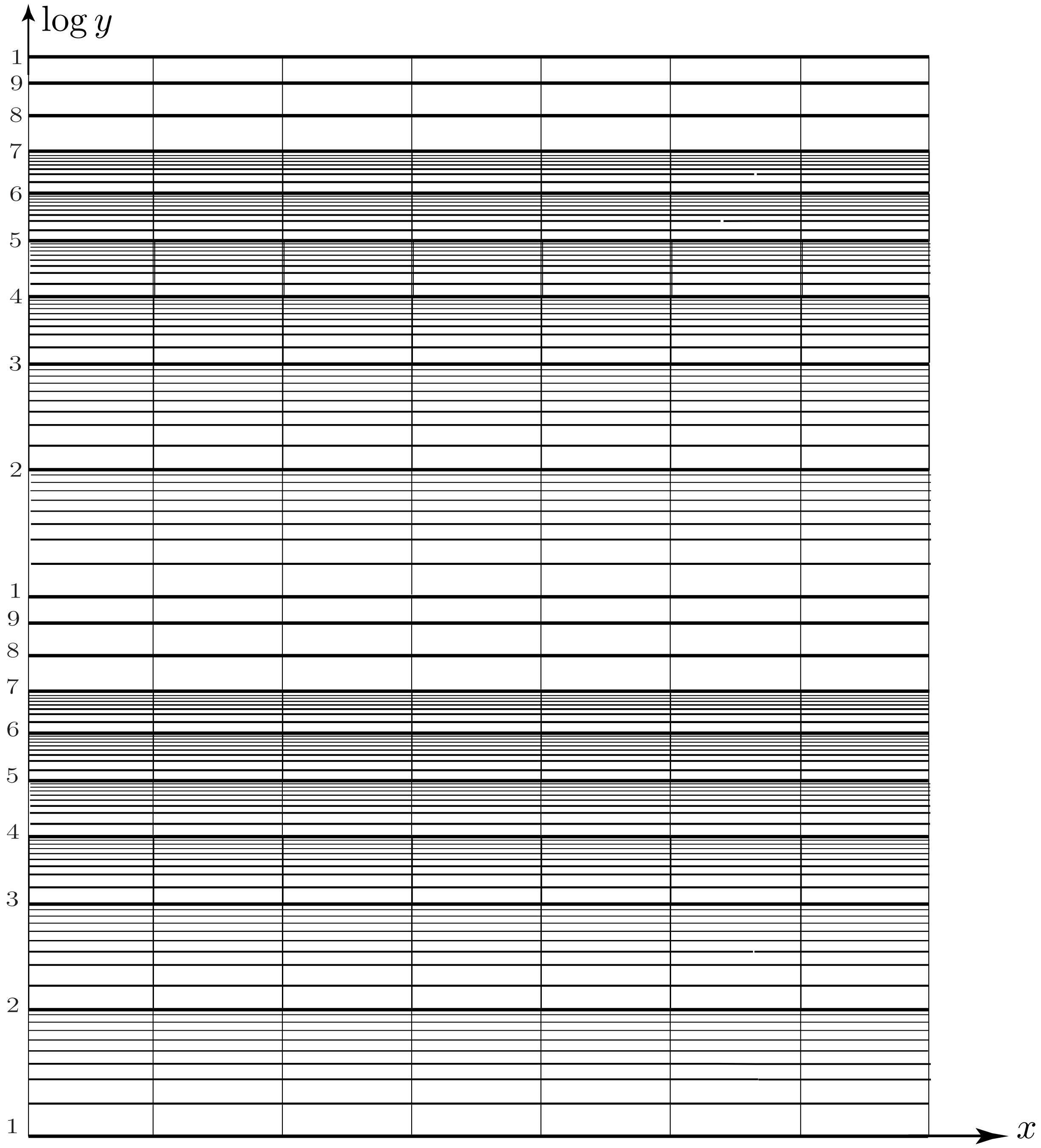
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