Modelling Exercises 6.5

Introduction

This Section provides examples and tasks employing exponential functions and logarithmic functions, such as growth and decay models which are important throughout science and engineering.

Prerequisites

Before starting this Section you should . . .

- be familiar with the laws of logarithms
- have knowledge of logarithms to base 10
- be able to solve equations involving logarithms and exponentials

Learning Outcomes

On completion you should be able to . . .

- develop exponential growth and decay models

1. Exponential increase

Task

(a) Look back at Section 6.2 to review the definitions of *an* exponential function and *the* exponential function.

(b) List examples in this Workbook of contexts in which exponential functions are appropriate.

Your solution

Answer

(a) *An* exponential function has the form $y = a^x$ where $a > 0$. *The* exponential function has the form $y = e^x$ where $e = 2.718282......$

(b) It is stated that exponential functions are useful when modelling the shape of a hanging chain or rope under the effect of gravity or for modelling exponential growth or decay.

We will look at a specific example of the exponential function used to model a population increase.

Task

Given that

$$P = 12e^{0.1t} \qquad (0 \leq t \leq 25)$$

where P is the number in the population of a city **in millions** at time t **in years** answer these questions.

(a) What does this model imply about P when $t = 0$?

(b) What is the stated upper limit of validity of the model?

(c) What does the model imply about values of P over time?

(d) What does the model predict for P when $t = 10$? Comment on this.

(d) What does the model predict for P when $t = 25$? Comment on this.

Your solution

(a)

(b)

(c)

(d)

(e)

Answer

(a) At $t = 0$, $P = 12$ which represents the initial population of 12 million. (Recall that $e^0 = 1$.)

(b) The time interval during which the model is valid is stated as $(0 \le t \le 25)$ so the model is intended to apply for 25 years.

(c) This is exponential growth so P will increase from 12 million at an accelerating rate.

(d) $P(10) = 12\mathrm{e}^1 \approx 33$ million. This is getting very large for a city but might be attainable in 10 years and just about sustainable.

(e) $P(25) = 12\mathrm{e}^{2.5} \approx 146$ million. This is unrealistic for a city.

Note that exponential population growth of the form $P = P_0e^{kt}$ means that as t becomes large and positive, P becomes very large. Normally such a population model would be used to predict values of P for $t > 0$, where $t = 0$ represents the present or some fixed time when the population is known. In Figure 6, values of P are shown for $t < 0$. These correspond to extrapolation of the model into the past. Note that as t becomes increasingly negative, P becomes very small but is never zero or negative because e^{kt} is positive for all values of t. The parameter k is called the **instantaneous fractional growth rate**.

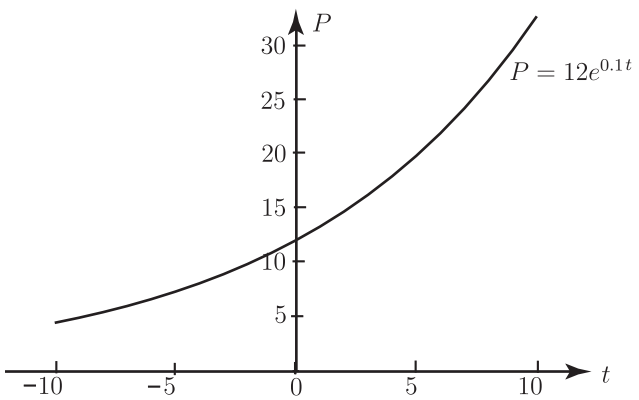


Figure 6: The function $P = 12e^{0.01t}$

For the model $P = 12\mathrm{e}^{kt}$ we see that $k = 0.1$ is unrealistic, and more realistic values would be $k = 0.01$ or $k = 0.02$. These would be similar but k=0.02 implies a faster growth for $t > 0$ than $k = 0.01$. This is clear in the graphs for $k = 0.01$ and $k = 0.02$ in Figure 7. The functions are plotted up to 200 years to emphasize the increasing difference as t increases.

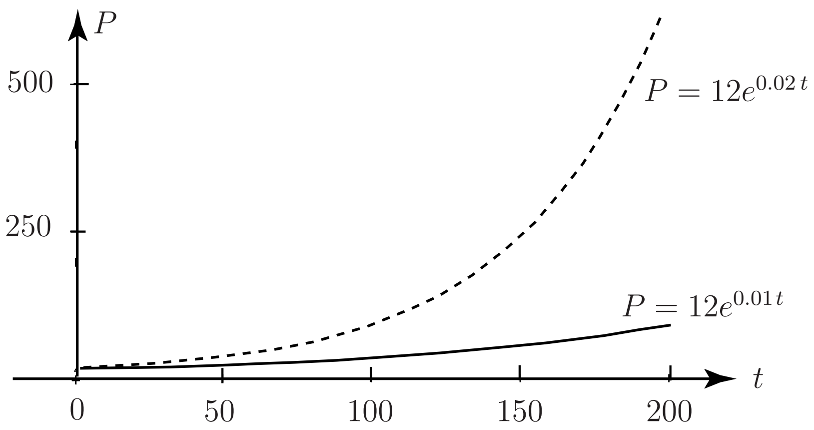


Figure 7: Comparison of the functions $P = 12e^{0.01t}$ and $P = 12e^{0.02t}$

The exponential function may be used in models for other types of growth as well as population growth. A general form may be written

$$y = ae^{bx} \qquad a > 0, \quad b > 0, \quad c \leq x \leq d$$

where a represents the value of y at $x = 0$. The value a is the intercept on the y-axis of a graphical representation of the function. The value b controls the rate of growth and c and d represent limits on x.

In the general form, a and b represent the **parameters** of the exponential function which can be selected to fit any given modelling situation where an exponential function is appropriate.

2. Linearisation of exponential functions

This subsection relates to the description of log-linear plots covered in Section 6.6.

Frequently in engineering, the question arises of how the parameters of an exponential function might be found from given data. The method follows from the fact that it is possible to 'undo' the exponential function and obtain a linear function by means of the logarithmic function. Before showing the implications of this method, it may be necessary to remind you of some rules for manipulating logarithms and exponentials. These are summarised in Table 1 on the next page, which exactly matches the general list provided in Key Point 8 in Section 6.3 (page 22.)

Table 1: Rules for manipulating base e logarithms and exponentials

Number	Rule	Number	Rule
1a	$\ln(xy) = \ln(x) + \ln(y)$	1b	$e^x \times e^y = e^{x+y}$
2a	$\ln(x/y) = \ln(x) - \ln(y)$	2b	$e^x / e^y = e^{x-y}$
3a	$\ln(x^y) = y\ln(x)$	3b	$(e^x)^y = e^{xy}$
4a	$\ln(e^x) = x$	4b	$e^{\ln(x)} = x$
5a	$\ln(e) = 1$	5b	$e^1 = e$
6a	$\ln(1) = 0$	6b	$e^0 = 1$

We will try 'undoing' the exponential in the particular example

$$P = 12e^{0.1t}$$

We take the natural logarithm ($\ln$) of both sides, which means logarithm to the base e. So

$$\ln(P) = \ln(12e^{0.1t})$$

The result of using Rule 1a in Table 1 is

$$\ln(P) = \ln(12) + \ln(e^{0.1t}).$$

The natural logarithmic functions 'undoes' the exponential function, so by Rule 4a,

$$\ln(e^{0.1t}) = 0.1t$$

and the original equation for P becomes

$$\ln(P) = \ln(12) + 0.1t.$$

Compare this with the general form of a linear function $y = ax + b$.

$$\begin{array}{ccccc} y & = & ax & + & b \\ \downarrow & & \downarrow & & \downarrow \\ \ln(P) & = & 0.1t & + & \ln(12) \end{array}$$

If we regard $\ln(P)$ as equivalent to y, 0.1 as equivalent to the constant a, t as equivalent to x, and $\ln(12)$ as equivalent to the constant b, then we can identify a linear relationship between $\ln(P)$ and t. A plot of $\ln(P)$ against t should result in a straight line, of slope 0.1, which crosses the $\ln(P)$ axis at $\ln(12)$. (Such a plot is called a **log-linear** or **log-lin** plot.) This is not particularly interesting here because we know the values 12 and 0.1 already.

Suppose, though, we want to try using the general form of the exponential function

$$P = ae^{bt} \qquad (c \leq t \leq d)$$

to create a continuous model for a population for which we have some discrete data. The first thing to do is to take logarithms of both sides

$$\ln(P) = \ln(ae^{bt}) \qquad (c \leq t \leq d).$$

Rule 1 from Table 1 then gives

$$\ln(P) = \ln(a) + \ln(e^{bt}) \qquad (c \leq t \leq d).$$

But, by Rule 4a, $\ln(e^{bt}) = bt$, so this means that

$$\ln(P) = \ln(a) + bt \qquad (c \leq t \leq d).$$

So, given some 'population versus time' data, for which you believe can be modelled by some version of the exponential function, plot the natural logarithm of population against time. If the exponential function is appropriate, the resulting data points should lie on or near a straight line. The slope of the straight line will give an estimate for b and the intercept with the $\ln(P)$ axis will give an estimate for $\ln(a)$. You will have carried out a **logarithmic transformation** of the original data for P. We say the original variation has been **linearised**.

A similar procedure will work also if any exponential function rather than the base e exponential function is used. For example, suppose that we try to use the function

$$P = A \times 2^{Bt} \qquad (C \leq t \leq D),$$

where A and B are constant parameters to be derived from the given data. We can take natural logarithms again to give

$$\ln(P) = \ln(A \times 2^{Bt}) \qquad (C \leq t \leq D).$$

Rule 1a from Table 1 then gives

$$\ln(P) = \ln(A) + \ln(2^{Bt}) \qquad (C \leq t \leq D).$$

Rule 3a then gives

$$\ln(2^{Bt}) = Bt\ln(2) = B\ln(2)\ t$$

and so

$$\ln(P) = \ln(A) + B\ln(2)\ t \qquad (C \leq t \leq D).$$

Again we have a straight line graph with the same intercept as before, $\ln A$, but this time with slope $B\ln(2)$.

Task

The amount of money $£M$ to which $£1$ grows after earning interest of 5% p.a. for N years is worked out as

$$M = 1.05^N$$

Find a linearised form of this equation.

Your solution

Answer

Take natural logarithms of both sides.

$$\ln(M) = \ln(1.05^N).$$

Rule 3b gives

$$\ln(M) = N\ln(1.05).$$

So a plot of $\ln(M)$ against N would be a straight line passing through $(0, 0)$ with slope $\ln(1.05)$.

The linearisation procedure also works if logarithms other than natural logarithms are used. We start again with

$$P = A \times 2^{Bt} \qquad (C \le t \le D).$$

and will take logarithms to base 10 instead of natural logarithms. Table 2 presents the laws of logarithms and indices (based on Key Point 8 page 22) interpreted for $\log_{10}$.

Table 2: Rules for manipulating base 10 logarithms and exponentials

Number	Rule	Number	Rule
1a	$\log_{10}(AB) = \log_{10} A + \log_{10} B$	1b	$10^A 10^B = 10^{A+B}$
2a	$\log_{10}(A/B) = \log_{10} A - \log_{10} B$	2b	$10^A/10^B = 10^{A-B}$
3a	$\log_{10}(A^k) = k \log_{10} A$	3b	$(10^A)^k = 10^{kA}$
4a	$\log_{10}(10^A) = A$	4b	$10^{\log_{10} A} = A$
5a	$\log_{10} 10 = 1$	5b	$10^1 = 10$
6a	$\log_{10} 1 = 0$	6b	$10^0 = 1$

Taking logs of $P = A \times 2^{Bt}$ gives:

$$\log_{10}(P) = \log_{10}(A \times 2^{Bt}) \qquad (C \le t \le D).$$

Rule 1a from Table 2 then gives

$$\log_{10}(P) = \log_{10}(A) + \log_{10}(2^{Bt}) \qquad (C \le t \le D).$$

Use of Rule 3a gives the result

$$\log_{10}(P) = \log_{10}(A) + B \log_{10}(2)\ t \qquad (C \le t \le D).$$

Task

(a) Write down the straight line function corresponding to taking logarithms of the general exponential function

$$P = ae^{bt} \qquad (c \le t \le d)$$

by taking logarithms to base 10.

(b) Write down the slope of this line.

Your solution

Answer

(a) $\log_{10}(P) = \log_{10}(a) + (b \log_{10}(e))t \qquad (c \le t \le d)$

(b) $b \log_{10}(e)$

It is not usually necessary to declare the subscript 10 when indicating logarithms to base 10. If you meet the term 'log' it will probably imply "to the base 10". In the remainder of this Section, the subscript 10 is dropped where $\log_{10}$ is implied.

3. Exponential decrease

Consider the value, £D, of a car subject to depreciation, in terms of the age A years of the car. The car was bought for £10500. The function

$$D = 10500e^{-0.25A} \qquad (0 \leq A \leq 6)$$

could be considered appropriate on the ground that (a) D had a fixed value of £10500 when $A = 0$, (b) D decreases as A increases and (c) D decreases faster when A is small than when A is large. A plot of this function is shown in Figure 8.

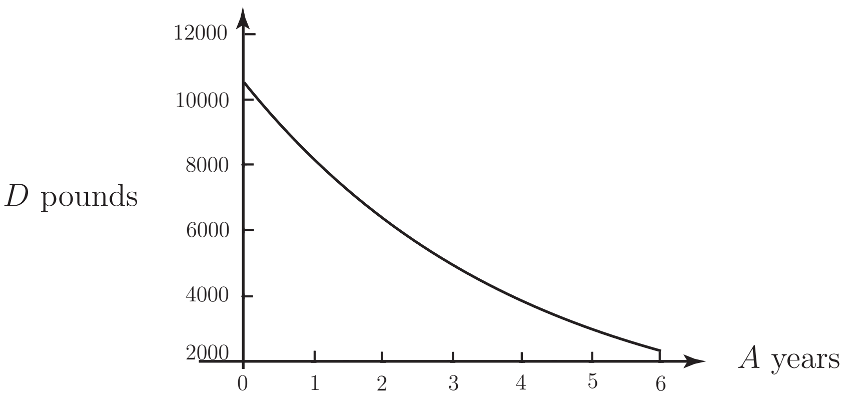


Figure 8: Plot of car depreciation over 6 years

Task

Produce the linearised model of $D = 10500e^{-0.25A}$.

Your solution

Answer

$\ln D = \ln 10500 + \ln(e^{-0.25A})$

so $\ln D = \ln 10500 - 0.25A$

Engineering Example 2

Exponential decay of sound intensity

Introduction

The rate at which a quantity decays is important in many branches of engineering and science. A particular example of this is exponential decay. Ideally the sound level in a room where there are substantial contributions from reflections at the walls, floor and ceiling will decay exponentially once the source of sound is stopped. The decay in the sound intensity is due to absorbtion of sound at the room surfaces and air absorption although the latter is significant only when the room is very large. The contributions from reflection are known as **reverberation**. A measurement of reverberation in a room of known volume and surface area can be used to indicate the amount of absorption.

Problem in words

As part of an emergency test of the acoustics of a concert hall during an orchestral rehearsal, consultants asked the principal trombone to play a single note at maximum volume. Once the sound had reached its maximum intensity the player stopped and the sound intensity was measured for the next 0.2 seconds at regular intervals of 0.02 seconds. The initial maximum intensity at time 0 was 1. The readings were as follows:

time	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
intensity	1	0.63	0.35	0.22	0.13	0.08	0.05	0.03	0.02	0.01	0.005

Draw a graph of intensity against time and, assuming that the relationship is exponential, find a function which expresses the relationship between intensity and time.

Mathematical statement of problem

If the relationship is exponential then it will be a function of the form

$$I = I_0 10^{kt}$$

and a log-linear graph of the values should lie on a straight line. Therefore we can plot the values and find the gradient and the intercept of the resulting straight-line graph in order to find the values for I_0 and k.

k is the gradient of the log-linear graph i.e.

$$k = \frac{\text{change in } \log_{10} \text{ (intensity)}}{\text{change in time}}$$

and I_0 is found from where the graph crosses the vertical axis $\log_{10}(I_0) = c$

Mathematical analysis

Figure 9(a) shows the graph of intensity against time.

We calculate the $\log_{10}$ (intensity) to create the table below:

time	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
$\log_{10}$(intensity)	0	-0.22	-0.46	-0.66	-0.89	-1.1	-1.3	-1.5	-1.7	-2.0	-2.2

Figure 9(b) shows the graph of $\log$ (intensity) against time.

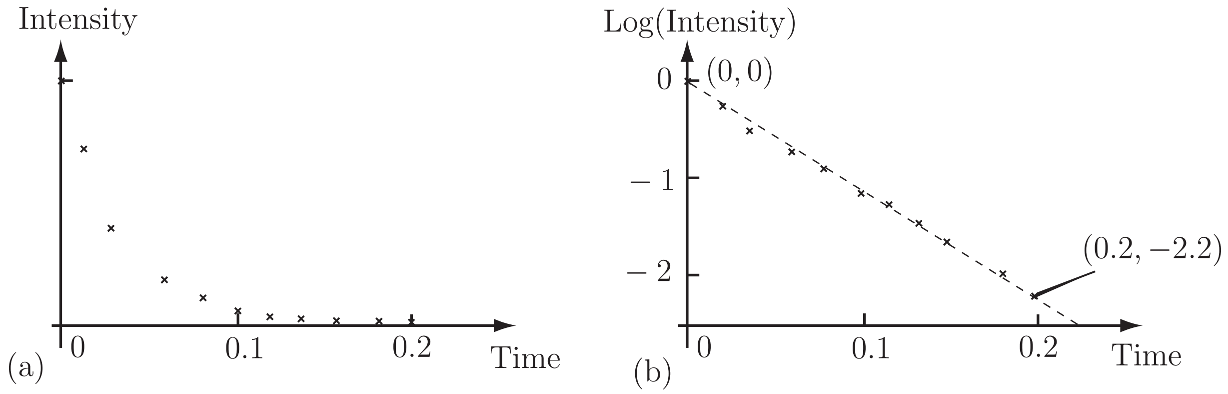


Figure 9: (a) Graph of sound intensity against time (b) Graph of $\log_{10}$ (intensity) against time and a line fitted by eye to the data. The line goes through the points $(0, 0)$ and $(0.2, -2.2)$.

We can see that the second graph is approximately a straight line and therefore we can assume that the relationship between the intensity and time is exponential and can be expressed as

$$I = I_0 10^{kt}.$$

The $\log_{10}$ of this gives

$$\log_{10}(I) = \log_{10}(I_0) + kt.$$

From the graph (b) we can measure the gradient, k using

$$k = \frac{\text{change in } \log_{10}\text{(intensity)}}{\text{change in time}}$$

giving $k = \dfrac{-2.2 - 0}{0.2 - 0} = -11$

The point at which it crosses the vertical axis gives

$$\log_{10}(I_0) = 0 \Rightarrow I_0 = 10^0 = 1$$

Therefore the expression $I = I_0 10^{kt}$ becomes

$$I = 10^{-11t}$$

Interpretation

The data recorded for the sound intensity fit exponential decaying with time. We have used a log-linear plot to obtain the approximate function:

$$I = 10^{-11t}$$

4. Growth and decay to a limit

Consider a function intended to represent the speed of a parachutist after the opening of the parachute where v m s^{-1} is the instantanous speed at time t s. An appropriate function is

$$v = 12 - 8e^{-1.25t} \qquad (t \geq 0),$$

We will look at some of the properties and modelling implications of this function. Consider first the value of v when $t = 0$:

$$v = 12 - 8e^0 = 12 - 8 = 4$$

This means the function predicts that the parachutist is moving at 4 m s^{-1} when the parachute opens. Consider next the value of v when t is arbitrarily large. For such a value of $t, 8e^{-1.25t}$ would be arbitrarily small, so v would be very close to the value 12. The modelling interpretation of this is that eventually the speed becomes very close to a constant value, 12 m s^{-1} which will be maintained until the parachutist lands.

The steady speed which is approached by the parachutist (or anything else falling against air resistance) is called the **terminal velocity**. The parachute, of course, is designed to ensure that the terminal velocity is sufficiently low (12 m s^{-1} in the specific case we have looked at here) to give a reasonably gentle landing and avoid injury.

Now consider what happens as t increases from near zero. When t is near zero, the speed will be near 4 m s^{-1}. The amount being subtracted from 12, through the term 8e$^{-1.25t}$, is close to 8 because $e^0 = 1$. As t increases the value of 8e$^{-1.25t}$ decreases fairly rapidly at first and then more gradually until v is very nearly 12. This is sketched in Figure 10. In fact v is never equal to 12 but gets imperceptibly close as anyone would like as t increases. The value shown as a horizontal broken line in Figure 10 is called an **asymptotic limit** for v.

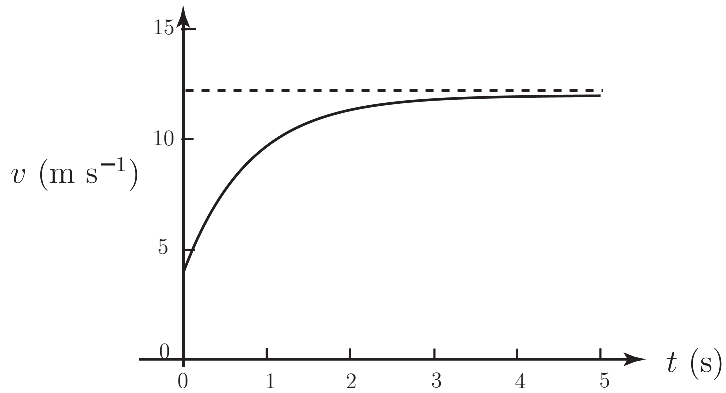


Figure 10: Graph of a parachutist's speed against time

The model concerned the approach of a parachutist's velocity to terminal velocity but the kind of behaviour portrayed by the resulting function is useful generally in modelling any **growth to a limit**. A general form of this type of growth-to-a-limit function is

$$y = a - be^{-kx} \qquad (C \leq x \leq D)$$

where a, b and k are positive constants (parameters) and C and D represent values of the independent variable between which the function is valid. We will now check on the properties of this general function. When $x = 0, y = a - be^0 = a - b$. As x increases the exponential factor e^{-kx} gets smaller, so y will increase from the value $a - b$ but at an ever-decreasing rate. As be^{-kx} becomes very small,

y, approaches the value a. This value represents the limit, towards which y grows. If a function of this general form was being used to create a model of population growth to a limit, then a would represent the limiting population, and $a - b$ would represent the starting population.

There are three parameters, a, b, and k in the general form. Knowledge of the initial and limiting population only gives two pieces of information. A value for the population at some non-zero time is needed also to evaluate the third parameter k.

As an example we will obtain a function to describe a food-limited bacterial culture that has 300 cells when first counted, has 600 cells after 30 minutes but seems to have approached a limit of 4000 cells after 18 hours.

We start by assuming the general form of growth-to-a-limit function for the bacteria population, with time measured in hours

$$P = a - be^{-kt} \qquad (0 \leq t \leq 18).$$

When $t = 0$ (the start of counting), $P = 300$. Since the general form gives $P = a - b$ when $t = 0$, this means that

$$a - b = 300.$$

The limit of P as t gets large, according to the general form $P = a - b^{-kt}$, is a, so $a = 4000$. From this and the value of $a - b$, we deduce that $b = 3700$. Finally, we use the information that $P = 600$ when t (measuring time in hours) $= 0.5$. Substitution in the general form gives

$$600 = 4000 - 3700e^{-0.5k}$$

$$3400 = 3700e^{-0.5k}$$

$$\frac{3400}{3700} = e^{-0.5k}$$

Taking natural logs of both sides:

$$\ln\left(\frac{3400}{3700}\right) = -0.5k \quad \text{so} \quad k = -2\ln(\frac{34}{37}) = 0.1691$$

Note, as a check, that k turns out to be positive as required for a growth-to-a-limit behaviour. Finally the required function may be written

$$P = 4000 - 3700e^{-0.1691t} \qquad (0 \leq t \leq 18).$$

As a check we should substitute $t = 18$ in this equation. The result is $P = 3824$ which is close to the required value of 4000.

Task

Find a function that could be used to model the growth of a population that has a value of 3000 when counts start, reaches a value of 6000 after 1 year but approaches a limit of 12000 after a period of 10 years.

(a) First find the modelling equation:

Your solution

Answer

Start with

$$P = a - be^{-kt} \qquad (0 \leq t \leq 10).$$

where P is the number of members of the population at time t years. The given data requires that a is 12000 and that $a - b = 3000$, so $b = 9000$.

The corresponding curve must pass through $(t = 1,\ P = 6000)$ so

$$6000 = 12000 - 9000e^{-k}$$

$$\mathrm{e}^{-k} = \frac{12000 - 6000}{9000} = \frac{2}{3} \qquad \text{so} \quad \mathrm{e}^{-kt} = (\mathrm{e}^{-k})^t = \left(\frac{2}{3}\right)^t$$ (using Rule 3b, Table 1, page 42)

So the population function is

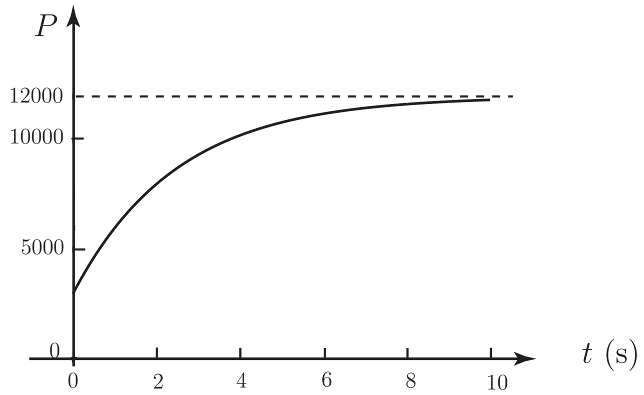
$$P = 12000 - 9000\left(\frac{2}{3}\right)^t \qquad (0 \leq t \leq 10).$$

Note that $P(10)$ according to this formula is approximately 11840, which is reasonably close to the required value of 12000.

(b) Now sketch this function:

Your solution

Answer



5. Inverse square law decay

Engineering Example 3

Inverse square law decay of electromagnetic power

Introduction

Engineers are concerned with using and intercepting many kinds of wave forms including electromagnetic, elastic and acoustic waves. In many situations the intensity of these signals decreases with the square of the distance. This is known as the **inverse square law**. The power received from a beacon antenna is expected to conform to the inverse square law with distance.

Problem in words

Check whether the data in the table below confirms that the measured power obeys this behaviour with distance.

Power received, W	0.393	0.092	0.042	0.021	0.013	0.008
Distance from antenna, m	1	2	3	4	5	6

Mathematical statement of problem

Represent power by P and distance by r. To show that the data fit the function $P = \dfrac{A}{r^2}$ where A is a constant, plot log(P) against log(r) (or plot the 'raw' data on log-log axes) and check

(a) how close the resulting graph is to that of a straight line

(b) how close the slope is to 2.

Mathematical analysis

The values corresponding to log(P) and log(r) are

log(P)	-0.428	-1.041	-1.399	-1.653	-1.851	-2.012
log(r)	0	0.301	0.499	0.602	0.694	0.778

These are plotted in Figure 11 and it is clear that they lie close to a straight line.

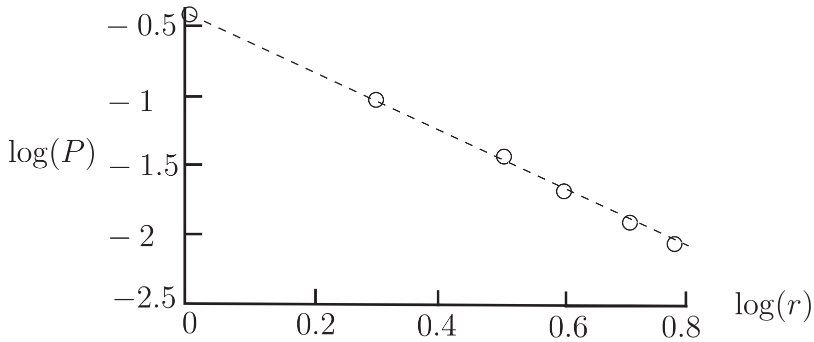


Figure 11

The slope of a line through the first and third points can be found from

$$\frac{-1.399 - (-0.428)}{0.499 - 0} = -2.035$$

The negative value means that the line slopes downwards for increasing r. It would have been possible to use any pair of points to obtain a suitable line but note that the last point is least 'in line' with the others. Taking logarithms of the equation $P = \dfrac{A}{r^n}$ gives $\log(P) = \log(A) - n\log(r)$

The inverse square law corresponds to $n = 2$. In this case the data yield $n = 2.035 \approx 2$. Where log(r) $= 0$, log(P) $=$ log(A). This means that the intercept of the line with the log(P) axis gives the value of log(A) $= -0.428$. So $A = 10 - 0.428 = 0.393$.

Interpretation

If the power decreases with distance according to the inverse square law, then the slope of the line should be -2. The calculated value of $n = 2.035$ is sufficiently close to confirm the inverse square law. The values of A and n calculated from the data imply that P varies with r according to

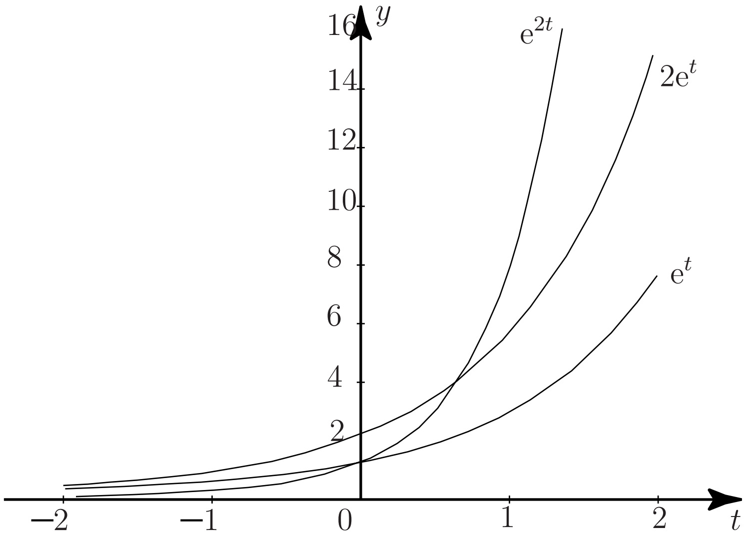
$$P = \frac{0.4}{r^2}$$

The slope of the line on a log-log plot is a little larger than -2. Moreover the points at 5 m and 6 m range fall below the line so there may be additional attenuation of the power with distance compared with predictions of the inverse square law.

Exercises

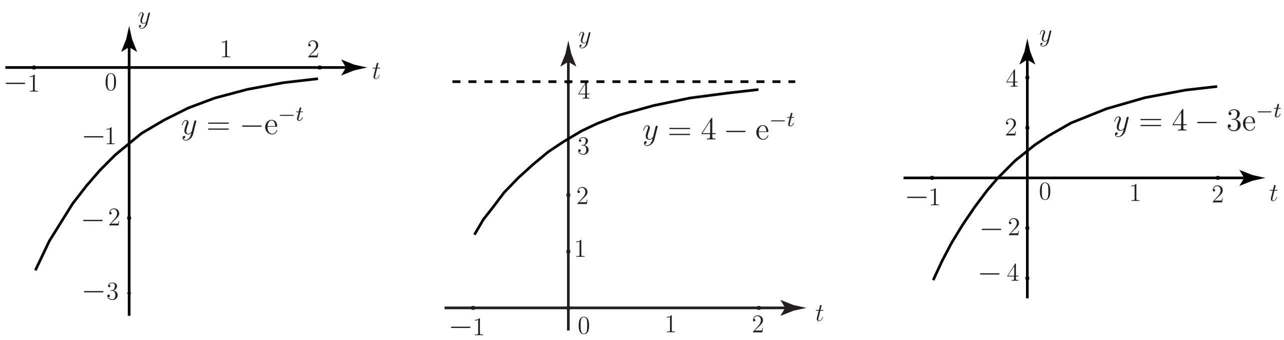
1. Sketch the graphs of (a) $y = \mathrm{e}^t$ (b) $y = \mathrm{e}^t + 3$ (c) $y = \mathrm{e}^{-t}$ (d) $y = \mathrm{e}^{-t} - 1$

2. The figure below shows the graphs of $y = \mathrm{e}^t$, $y = 2\mathrm{e}^t$ and $y = \mathrm{e}^{2t}$.



State in words how the graphs of $y = 2\mathrm{e}^t$ and $y = \mathrm{e}^{2t}$ relate to the graph of $y = \mathrm{e}^t$.

3. The figures below show graphs of $y = -\mathrm{e}^{-t}$, $y = 4 - \mathrm{e}^{-t}$ and $y = 4 - 3\mathrm{e}^{-t}$.



Use the above graphs to help you to sketch graphs of (a) $y = 5 - \mathrm{e}^{-t}$ (b) $y = 5 - 2\mathrm{e}^{-t}$

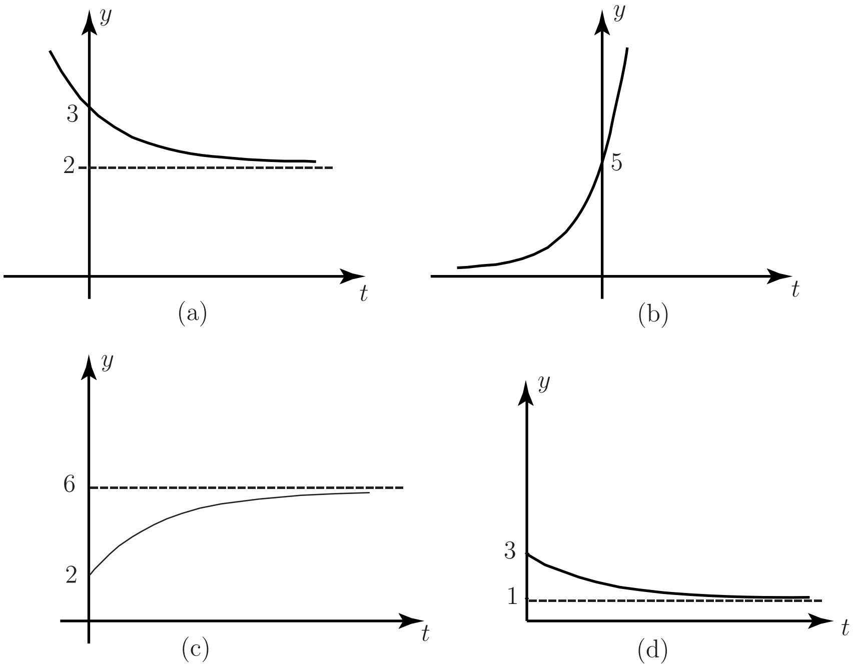
4. (a) The graph (a) in the figure below has an equation of the form

   $y = A + \mathrm{e}^{-kt}$, where A and k are constants. What is the value of A?

   (b) The graph (b) below has an equation of the form $y = A\mathrm{e}^{kt}$ where A and k are constants. What is the value of A?

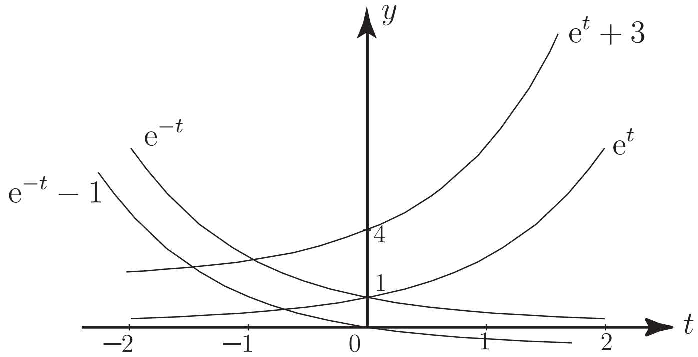
   (c) Write down a possible form of the equation of the exponential graph (c) giving numerical values to as many constants as possible.

   (d) Write down a possible form of the equation of the exponential graph (d) giving numerical values to as many constants as possible.



Answers

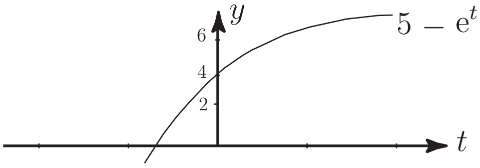
1.



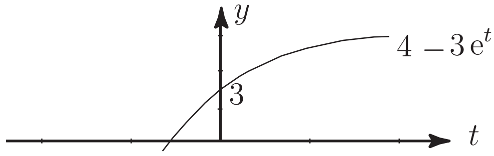
2. (a) $y = 2e^t$ is the same shape as $y = e^t$ but with all y values doubled.

   (b) $y = e^{2t}$ is much steeper than $y = e^t$ for $t > 0$ and much flatter for $t < 0$. Both pass through $(0, 1)$. Note that $y = e^{2t} = (e^t)^2$ so each value of $y = e^{2t}$ is the square of the corresponding value of $y = e^t$.

3. (a)



(b)



4. (a) 2 (b) 5 (c) $y = 6 - 4e^{-kt}$ (d) $y = 1 + 2e^{-kt}$

6. Logarithmic relationships

Experimental psychology is concerned with observing and measuring human response to various stimuli. In particular, sensations of light, colour, sound, taste, touch and muscular tension are produced when an external stimulus acts on the associated sense. A nineteenth century German, Ernst Weber, conducted experiments involving sensations of heat, light and sound and associated stimuli. Weber measured the response of subjects, in a laboratory setting, to input stimuli measured in terms of energy or some other physical attribute and discovered that:

(1) No sensation is felt until the stimulus reaches a certain value, known as the threshold value.

(2) After this threshold is reached an increase in stimulus produces an increase in sensation.

(3) This increase in sensation occurs at a diminishing rate as the stimulus is increased.

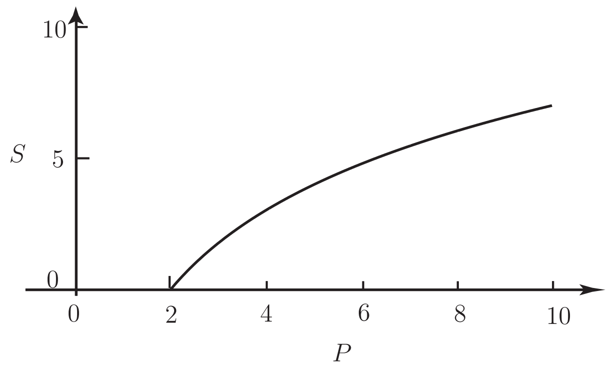
Task

(a) Do Weber's results suggest a linear or non-linear relationship between sensation and stimulus? Sketch a graph of sensation against stimulus according to Weber's results.

(b) Consider whether an exponential function or a growth-to-a-limit function might be an appropriate model.

Answer

(a) Non-linearity is required by observation (3).



(b) An exponential-type of growth is not appropriate for a model consistent with these experimental results, since we need a diminishing rate of growth in sensation as the stimulus increases. A growth-to-a-limit type of function is not appropriate since the data, at least over the range of Weber's experiments, do not suggest that there is a limit to the sensation with continuing increase in stimulus; only that the increase in sensation occurs more and more slowly.

A late nineteenth century German scientist, Gustav Fechner, studied Weber's results. Fechner suggested that an appropriate function modelling Weber's findings would be logarithmic. He suggested that the variation in sensation (S) with the stimulus input (P) is modelled by

$$S = A\log(P/T) \qquad (0 < T \leq 1)$$

where T represents the threshold of stimulus input below which there is no sensation and A is a constant. Note that when $P = T, \log(P/T) = \log(1) = 0$, so this function is consistent with item (1) of Weber's results. Recall also that $\log$ means logarithm to base 10, so when $P = 10T, S = A\log(10) = A$. When P = 100T, $S = A\log(100) = 2A$. The logarithmic function predicts that a tenfold increase in the stimulus input from T to $10T$ will result in the same change in sensation as a further tenfold increase in stimulus input to $100T$. Each tenfold change is stimulus results in a doubling of sensation. So, although sensation is predicted to increase with stimulus, the stimulus has to increase at a faster and faster rate (i.e. exponentially) to achieve a given change in sensation. These points are consistent with items (2) and (3) of Weber's findings. Fechner's suggestion, that the logarithmic function is an appropriate one for a model of the relationship between sensation and stimulus, seems reasonable. Note that the logarithmic function suggested by Weber is not defined for zero stimulus but we are only interested in the model at and above the threshold stimulus, i.e. for values of the logarithm equal to and above zero. Note also that the logarithmic function is useful for looking at changes in sensation relative to stimulus values other than the threshold stimulus. According to Rule 2a in Table 2 on page 42, Fechner's sensation function may be written

$$S = A\log(P/T) = A[\log(P) - \log(T)] \qquad (P \geq T > 0).$$

Suppose that the sensation has the value S_1 at P_1 and S_2 at P_2, so that

$$S_1 = A[\log(P_1) - \log(T)] \qquad (P_1 \geq T > 0),$$

and

$$S_2 = A[\log(P_2) - \log(T)] \qquad (P_2 \geq T > 0).$$

If we subtract the first of these two equations from the second, we get

$$S_2 - S_1 = A[\log(P_2) - \log(P_1)] = A\log(P_2/P_1),$$

where Rule 2a of Table 2 has been used again for the last step. According to this form of equation, the change in sensation between two stimuli values depends on the ratio of the stimuli values.
We start with

$$S = A\log(P/T) \qquad (1 \geq T > 0).$$

Divide both sides by A:

$$\frac{S}{A} = \log\frac{P}{T} \qquad (1 \geq T > 0).$$

'Undo' the logarithm on both sides by raising 10 to the power of each side:

$$10^{S/A} = 10^{\log(P/T)} = \frac{P}{T} \qquad (1 \geq T > 0), \text{ using Rule 4b of Table 2.}$$

So $P = T \times 10^{S/A} \qquad (1 \geq T > 0)$ which is an exponential relationship between stimulus and sensation.

A **logarithmic** relationship between sensation and stimulus therefore implies an **exponential** relationship between stimulus and sensation. The relationship may be written in two different forms with the variables playing opposite roles in the two functions.

The logarithmic relationship between sensation and stimulus is known as the *Weber-Fechner Law of Sensation.* The idea that a mathematical function could describe our sensations was startling when first propounded. Indeed it may seem quite amazing to you now. Moreover it doesn't always work. Nevertheless the idea has been quite fruitful. Out of it has come much quantitative experimental psychology of interest to sound engineers. For example, it relates to the sensation of the loudness of sound. Sound level is expressed on a logarithmic scale. At a frequency of 1 kHz an increase of 10 dB corresponds to a doubling of loudness.

Task

Given a relationship between y and x of the form $y = 3\log(\dfrac{x}{4}) \qquad (x \geq 4)$, find the relationship between x and y.

Your solution

Answer

One way of answering is to compare with the example preceding this task. We have y in place of S, x in place of P, 3 in place of A, 4 in place of T. So it is possible to write down immediately

$$x = 4 \times 10^{y/3} \qquad (y \geq 0)$$

Alternatively we can manipulate the given expression algebraically.

Starting with $y = 3\log(x/4)$, divide both sides by 3 to give $y/3 = \log(x/4)$.

Raise 10 to the power of each side to eliminate the log, so that $10^{y/3} = x/4$.

Multiply both sides by 4 and rearrange, to obtain $x = 4 \times 10^{y/3}$, as before.

The associated range is the result of the fact that $x \geq 4$, so $10^{y/3} \geq 1$, so $y/3 > 0$ which means $y > 0$.