The Logarithmic Function

6.4

Introduction

In this Section we consider the logarithmic function $y = \log_a x$ and examine its important characteristics. We see that this function is only defined if x is a positive number. We also see that the $\log$ function is the inverse of the exponential function and vice versa. We show, through numerous examples, how equations involving logarithms and exponentials can be solved.

Prerequisites

Before starting this Section you should ...

- have knowledge of inverse functions
- have knowledge of the laws of logarithms and of the laws of indices
- be able to solve quadratic equations

Learning Outcomes

On completion you should be able to ...

- explain the relation between the logarithm and the exponential function
- solve equations involving exponentials and logarithms

1. The logarithmic function

In Section 6.3 we introduced the operation of taking logarithms which reverses the operation of exponentiation.

If $a > 0$ and $a \neq 1$ then $\quad x = a^y \quad$ implies $\quad y = \log_a x$

In this Section we consider the log function in more detail. We shall concentrate only on the functions $\log x$ (i.e. to base 10) and $\ln x$ (i.e. to base e). The functions $y = \log x$ and $y = \ln x$ have similar characteristics. We can never choose x as a negative number since 10^y and e^y are each always positive. The graphs of $y = \log x$ and $y = \ln x$ are shown in Figure 5.

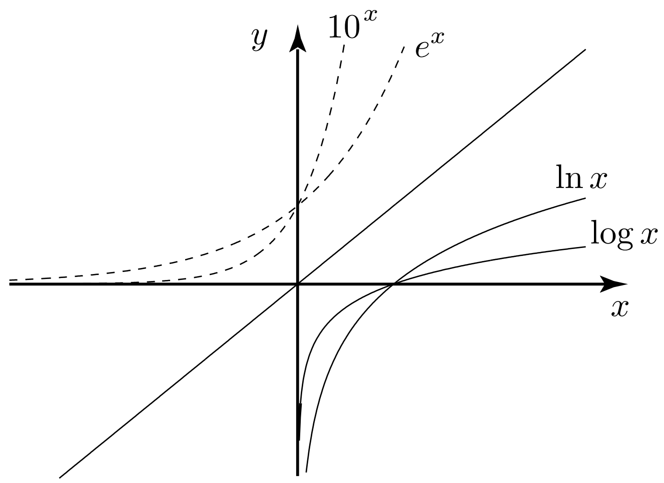

Figure 5: Logarithmic and exponential functions

From the graphs we see that both functions are one-to-one so each has an **inverse function** - the inverse function of $\log_a x$ is a^x. Let us do this for logs to base 10.

2. Solving equations involving logarithms and exponentials

To solve equations which involve logarithms or exponentials we need to be aware of the basic laws which govern both of these mathematical concepts. We illustrate by considering some examples.

Example 6

Solve for the variable x: $\quad$ (a) $3 = 10^x$, $\quad$ (b) $10^{x/4} = \log 3$, $\quad$ (c) $\dfrac{1}{17 - \mathrm{e}^x} = 4$

Solution

(a) Here we take logs (to base 10 because of the term 10^x) of both sides to get

$$\log 3 = \log 10^x = x \log 10 = x$$

where we have used the general property that $\log_a A^k = k \log_a A$ and the specific property that $\log 10 = 1$. Hence $x = \log 3$ or, in numerical form, $x = 0.47712$ to 5 d.p.

Solution (contd.)

(b) The approach used in (a) is used here. Take logs of both sides: $\log(10^{x/4}) = \log(\log 3)$

that is $\quad \dfrac{x}{4}\log 10 = \log(\log 3) = \log(0.4771212) = -0.3213712$

So, since $\log 10 = 1$, we have $\quad x = 4(-0.3213712) = -1.28549 \quad$ to 5 d.p.

(c) Here we simplify the expression before taking logs.

$$\frac{1}{17 - \mathrm{e}^x} = 4 \qquad \text{implies} \qquad 1 = 4(17 - \mathrm{e}^x)$$

or $\;4\mathrm{e}^x = 4(17) - 1 = 67\;$ so $\;\mathrm{e}^x = 16.75$. Now taking natural logs of both sides (because of the presence of the e^x term) we have:

$$\ln(\mathrm{e}^x) = \ln(16.75) = 2.8183983$$

But $\ln(\mathrm{e}^x) = x\ln \mathrm{e} = x$ and so the solution to $\dfrac{1}{17 - \mathrm{e}^x} = 4$ is $x = 2.81840$ to 5 d.p.

Task

Solve the equation $(\mathrm{e}^x)^2 = 50$

First solve for e^x by taking square roots of both sides:

Your solution

$(\mathrm{e}^x)^2 = 50 \quad$ implies $\quad \mathrm{e}^x =$

Answer

$(\mathrm{e}^x)^2 = 50 \quad$ implies $\quad \mathrm{e}^x = \sqrt{50} = 7.071068$. Here we have taken the positive value for the square root since we know that exponential functions **are always positive**.

Now take logarithms to an appropriate base to find x:

Your solution

$\mathrm{e}^x = 7.071068 \quad$ implies $\quad x =$

Answer

$\mathrm{e}^x = 7.071068 \quad$ implies $\quad x = \ln(7.071068) = 1.95601$ to 5 d.p.

Task

Solve the equation $e^{2x} = 17e^x$

First simplify the expression as much as possible (divide both sides by e^x):

Your solution

$$e^{2x} = 17e^x \quad \text{implies} \quad \frac{e^{2x}}{e^x} = 17 \quad \text{so}$$

Answer

$\dfrac{e^{2x}}{e^x} = 17$ implies $e^{2x-x} = 17$ so $e^x = 17$

Now complete the solution for x:

Your solution

$$e^x = 17 \quad \text{implies} \quad x =$$

Answer

$x = \ln(17) = 2.8332133$

Example 7

Find x if $\quad 10^x - 5 + 6(10^{-x}) = 0$

Solution

We first simplify this expression by multiplying through by 10^x (to eliminate the term 10^{-x}):

$$10^x(10^x) - 10^x(5) + 10^x(6(10^{-x})) = 0$$

or

$$(10^x)^2 - 5(10^x) + 6 = 0 \qquad \text{since} \qquad 10^x(10^{-x}) = 10^0 = 1$$

We realise that this expression is a **quadratic equation**. Let us put $y = 10^x$ to give

$$y^2 - 5y + 6 = 0$$

Now, we can factorise to give

$$(y-3)(y-2) = 0 \qquad \text{so that} \qquad y = 3 \quad \text{or} \quad y = 2$$

For each of these values of y we obtain a separate value for x since $y = 10^x$.

Case 1 If $y = 3$ then $3 = 10^x$ implying $x = \log 3 = 0.4771212$

Case 2 If $y = 2$ then $2 = 10^x$ implying $x = \log 2 = 0.3010300$

We conclude that the equation $10^x - 5 + 6(10^{-x}) = 0$ has two possible solutions for x: either $x = 0.4771212$ or $x = 0.3010300$, to 7 d.p.

Task

Solve $2e^{2x} - 7e^{x} + 3 = 0.$

First write this equation as a quadratic in the variable $y = e^x$ remembering that $e^{2x} \equiv (e^x)^2$:

Your solution

If $y = e^x$ then $2e^{2x} - 7e^x + 3 = 0$ becomes

Answer

$2y^2 - 7y + 3 = 0$

Now solve the quadratic for y:

Your solution

$2y^2 - 7y + 3 = 0$ implies $(2y \qquad)(y \qquad) = 0$

Answer

$(2y - 1)(y - 3) = 0$ therefore $y = \frac{1}{2}$ or $y = 3$

Finally, for each of your values of y, find x:

Your solution

If $y = \frac{1}{2}$ then $\frac{1}{2} = e^x$ implies $x =$

If $y = 3$ then $3 = e^x$ implies $x =$

Answer

$x = -0.693147$ or $x = 1.0986123$

Task

The temperature T , in degrees C, of a chemical reaction is given by the formula

$T = 80e^{0.03t} \times \ t \geq 0$, where t is the time, in seconds.

Calculate the time taken for the temperature to reach $150°$ C .

Answer

$$150 = 80e^{0.03t} \quad \Rightarrow \quad 1.875 = e^{0.03t} \quad \Rightarrow \quad \ln(1.875) = 0.03t \quad \Rightarrow \quad t = \frac{\ln(1.875)}{0.03}$$

This gives $t = 20.95$ to 2 d.p.

So the time is 21 seconds.

Engineering Example 1

Arrhenius' law

Introduction

Chemical reactions are very sensitive to temperature; normally, the rate of reaction increases as temperature increases. For example, the corrosion of iron and the spoiling of food are more rapid at higher temperatures. Chemically, the probability of collision between two molecules increases with temperature, and an increased collision rate results in higher kinetic energy, thus increasing the proportion of molecules that have the **activation energy** for the reaction, i.e. the minimum energy required for a reaction to occur. Based upon his observations, the Swedish chemist, Svante Arrhenius, proposed that the rate of a chemical reaction increases exponentially with temperature. This relationship, now known as Arrhenius' law, is written as

$$k = k_0 \exp\left(\frac{-E_a}{RT}\right) \qquad (1)$$

where k is the reaction rate constant, k_0 is the frequency factor, E_a is the activation energy, R is the universal gas constant and T is the absolute temperature. Thus, the reaction rate constant, k, depends on the quantities k_0 and E_a, which characterise a given reaction, and are generally assumed to be temperature independent.

Problem in words

In a laboratory, ethyl acetate is reacted with sodium hydroxide to investigate the reaction kinetics. Calculate the frequency factor and activation energy of the reaction from Arrhenius' Law, using the experimental measurements of temperature and reaction rate constant in the table:

T	310	350
k	7.757192	110.9601

Mathematical statement of problem

Given that $k = 7.757192$ s^{-1} at $T = 310$ K and $k = 110.9601$ s^{-1} at $T = 350$ K, use Equation (1) to produce two linear equations in E_a and k_0. Solve these to find E_a and k_0. (Assume that the gas constant $R = 8.314$ J K^{-1} mol^{-1}.)

Mathematical analysis

Taking the natural logarithm of both sides of (1)

$$\ln k = \ln\left\{k_0 \exp\left(\frac{-E_a}{RT}\right)\right\} = \ln k_0 - \frac{E_a}{RT}$$

Now inserting the experimental data gives the two linear equations in E_a and k_0

$$\ln k_1 = \ln k_0 - \frac{E_a}{RT_1} \qquad (2)$$

$$\ln k_2 = \ln k_0 - \frac{E_a}{RT_2} \qquad (3)$$

where $k_1 = 7.757192$, $T_1 = 310$ and $k_2 = 110.9601$, $T_2 = 350$.

Firstly, to find E_a, subtract Equation (2) from Equation (3)

$$\ln k_2 - \ln k_1 = \frac{E_a}{RT_1} - \frac{E_a}{RT_2} = \frac{E_a}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$

so that

$$E_a = \frac{R(\ln k_2 - \ln k_1)}{\left(\frac{1}{T_1} - \frac{1}{T_2}\right)}$$

and substituting the values gives

$$E_a = 60000 \text{ J mol}^{-1} = 60 \text{ kJ mol}^{-1}$$

Secondly, to find k_0, from (2)

$$\ln k_0 = \ln k_1 + \frac{E_a}{RT_1} \qquad \Rightarrow \qquad k_0 = \exp\left(\ln k_1 + \frac{E_a}{RT_1}\right) = k_1 \exp\left(\frac{E_a}{RT_1}\right)$$

and substituting the values gives

$$k_0 = 1.0 \times 10^{11} \text{ s}^{-1}$$

Task

The reaction

$$2NO_2(g) \longrightarrow 2NO(g) + O_2(g)$$

has a reaction rate constant of 1.0×10^{-10} s^{-1} at 300 K and activation energy of 111 kJ mol^{-1} = 111 000 J mol^{-1}. Use Arrhenius' law to find the reaction rate constant at a temperature of 273 K.

Your solution

Answer

Rearranging Arrhenius' equation gives

$$k_0 = k\,\exp\left(\frac{E_a}{RT}\right)$$

Substituting the values gives $k_0 = 2.126 \times 10^9$ s^{-1}

Now we use this value of k_0 with E_a in Arrhenius' equation (1) to find k at $T = 273$ K

$$k = k_0\,\exp\left(\frac{-E_a}{RT}\right) = 1.226 \times 10^{-12}\ \text{s}^{-1}$$

Task

For a chemical reaction with frequency factor $k_0 = 0.5$ s^{-1} and ratio $E_a/R = 800$ K, use Arrhenius' law to find the temperature at which the reaction rate constant would be equal to 0.1 s^{-1}.

Your solution

Answer

Rearranging Equation (1)

$$\frac{k}{k_0} = \exp\left(\frac{-E_a}{RT}\right)$$

Taking the natural logarithm of both sides

$$\ln\left(\frac{k}{k_0}\right) = \frac{-E_a}{RT}$$

so that

$$T = \frac{-E_a}{R\,\ln(k/k_0)} = \frac{E_a}{R\,\ln(k_0/k)}$$

Substituting the values gives $T = 497$ K

As a final example we consider equations involving the hyperbolic functions.

Example 8

Solve the equations

(a) $\cosh 3x = 1$ (b) $\cosh 3x = 2$ (c) $2\cosh^2 x = 3\cosh 2x - 3$

Solution

(a) From its graph we know that $\cosh x = 0$ only when $x = 0$, so we need $3x = 0$ which implies $x = 0$.

(b) $\cosh 3x = 2$ implies $\dfrac{e^{3x} + e^{-3x}}{2} = 2$ or $e^{3x} + e^{-3x} - 4 = 0$

Now multiply through by e^{3x} (to eliminate the term e^{-3x}) to give

$$e^{3x}e^{3x} + e^{3x}e^{-3x} - 4e^{3x} = 0 \qquad \text{or} \qquad (e^{3x})^2 - 4e^{3x} + 1 = 0$$

This is a quadratic equation in the variable e^{3x} so substituting $y = e^{3x}$ gives

$$y^2 - 4y + 1 = 0 \qquad \text{implying} \qquad y = 2 \pm \sqrt{3} \quad \text{so} \quad y = 3.7321 \quad \text{or} \quad 0.26795$$

$$e^{3x} = 3.7321 \qquad \text{implies} \qquad x = \frac{1}{3}\ln 3.7321 = 0.439 \quad \text{to 3 d.p.}$$

$$e^{3x} = 0.26795 \qquad \text{implies} \qquad x = \frac{1}{3}\ln 0.26795 = -0.439 \quad \text{to 3 d.p.}$$

(c) We first simplify this expression by using the identity: $\cosh 2x = 2\cosh^2 - 1$. Thus the original equation $2\cosh^2 x = 3\cosh 2x - 3$ becomes $\cosh 2x + 1 = 3\cosh 2x - 3$ or, when written in terms of exponentials:

$$\frac{e^{2x} + e^{-2x}}{2} = 3\left(\frac{e^{2x} + e^{-2x}}{2}\right) - 4$$

Multiplying through by $2e^{2x}$ gives $e^{4x} + 1 = 3(e^{4x} + 1) - 8e^{2x}$ or, after simplifying:

$$e^{4x} - 4e^{2x} + 1 = 0$$

Writing $y = e^{2x}$ we easily obtain $y^2 - 4y + 1 = 0$ with solution (using the quadratic formula):

$$y = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

If $y = 2 + \sqrt{3}$ then $2 + \sqrt{3} = e^{2x}$ implying $x = 0.65848$ to 5 d.p.

If $y = 2 - \sqrt{3}$ then $2 - \sqrt{3} = e^{2x}$ implying $x = -0.65848$ to 5 d.p.

Find the solution for x if $\tanh x = 0.5$.

First re-write $\tanh x$ in terms of exponentials:

Your solution

$\tanh x =$

Answer

$$\tanh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}} = \frac{\mathrm{e}^{2x} - 1}{\mathrm{e}^{2x} + 1}$$

Now substitute into $\tanh x = 0.5$:

Your solution

$\tanh x = 0.5$ implies $\dfrac{\mathrm{e}^{2x} - 1}{\mathrm{e}^{2x} + 1} = 0.5$ so, on simplifying, $\mathrm{e}^{2x} =$

Answer

$\dfrac{\mathrm{e}^{2x} - 1}{\mathrm{e}^{2x} + 1} = 0.5$ implies $(\mathrm{e}^{2x} - 1) = \dfrac{1}{2}(\mathrm{e}^{2x} + 1)$ so $\dfrac{\mathrm{e}^{2x}}{2} = \dfrac{3}{2}$ so, finally, $\mathrm{e}^{2x} = 3$

Now complete your solution by finding x:

Your solution

$\mathrm{e}^{2x} = 3$ so $x =$

Answer

$$x = \frac{1}{2}\ln 3 = 0.549306$$

Alternatively, many calculators can directly calculate the inverse function $\tanh^{-1}$. If you have such a calculator then you can use the fact that

$\tanh x = 0.5$ implies $x = \tanh^{-1} 0.5$ to obtain directly $x = 0.549306$

Example 9

Solve for x if $3\ln x + 4\log x = 1$.

Solution

This has logs to two different bases. So we must first express each logarithm in terms of logs to the *same* base, e say. From Key Point 8

$$\log x = \frac{\ln x}{\ln 10}$$

So $3\ln x + 4\log x = 1$ becomes

$$3\ln x + 4\frac{\ln x}{\ln 10} = 1 \qquad \text{or} \qquad (3 + \frac{4}{\ln 10})\ln x = 1$$

leading to $\ln x = \dfrac{\ln 10}{3\ln 10 + 4} = \dfrac{2.302585}{10.907755} = 0.211096$ and so

$$x = \mathrm{e}^{0.211096} = 1.2350311$$

Exercises

1. Solve for the variable x: (a) $\pi = 10^x$ (b) $10^{-x/2} = 3$ (c) $\dfrac{1}{17 - \pi^x} = 4$

2. Solve the equations

   (a) $\mathrm{e}^{2x} = 17\mathrm{e}^x$, (b) $\mathrm{e}^{2x} - 2\mathrm{e}^x - 6 = 0$, (c) $\cosh x = 3$.

Answers

1. (a) $x = \log\pi = 0.497$

   (b) $-x/2 = \log 3$ and so $x = -2\log 3 = -0.954$

   (c) $17 - \pi^x = 0.25$ so $\pi^x = 16.75$ therefore $x = \dfrac{\log 16.75}{\log\pi} = \dfrac{1.224}{0.497} = 2.462$

2. (a) Take logs of both sides: $2x = \ln 17 + x \qquad \therefore \quad x = \ln 17 = 2.833$

   (b) Let $y = \mathrm{e}^x$ then $y^2 - 2y - 6 = 0$ therefore $y = 1 \pm \sqrt{7}$ (we cannot take the negative sign since exponentials can never be negative). Thus $x = \ln(1 + \sqrt{7}) = 1.2936$.

   (c) $\mathrm{e}^x + \mathrm{e}^{-x} = 6$ therefore $\mathrm{e}^{2x} - 6\mathrm{e}^x + 1 = 0$ so $\mathrm{e}^x = \dfrac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm \sqrt{8}$

   We have, finally $x = \ln(3 + \sqrt{8}) = 1.7627$ or $x = \ln(3 - \sqrt{8}) = -1.7627$