Logarithms

6.3

Introduction

In this Section we introduce the logarithm: $\log_a b$. The operation of taking a logarithm essentially reverses the operation of raising a number to a power. We will formulate the basic laws satisfied by all logarithms and learn how to manipulate expressions involving logarithms. We shall see that to every law of indices there is an equivalent law of logarithms. Although logarithms to any positive base are defined it is common practice to employ only two kinds of logarithms: logs to base 10 and logs to base e.

Prerequisites

Before starting this Section you should ...

- have a knowledge of exponents and of the laws of indices

Learning Outcomes

On completion you should be able to ...

- invert $b = a^n$ using logarithms
- simplify expressions involving logarithms
- change bases in logarithms

1. Logarithms

Logarithms reverse the process of raising a base 'a' to a power 'n'. As with all exponentials, the base should be a positive number.

If $b = a^n$ then we write $\log_a b = n$.

Of course, the reverse statement is equivalent

If $\log_a b = n$ then $b = a^n$

The expression $\log_a b = n$ is read

"The log to base a of the number b is equal to n"

The term "log" is short for the word **logarithm**.

Example 3

Determine the log equivalents of
(a) $16 = 2^4$, (b) $16 = 4^2$, (c) $1000 = 10^3$,
(d) $134.896 = 10^{2.13}$, (e) $8.414867 = e^{2.13}$

Solution

(a) Since $16 = 2^4$ then $\log_2 16 = 4$
(b) Since $16 = 4^2$ then $\log_4 16 = 2$
(c) Since $1000 = 10^3$ then $\log_{10} 1000 = 3$
(d) Since $134.896 = 10^{2.13}$ then $\log_{10} 134.896 = 2.13$
(e) Since $8.41467 = \mathrm{e}^{2.13}$ then $\log_\mathrm{e} 8.414867 = 2.13$

Key Point 7

If	$b = a^n$	then	$\log_a b = n$
If	$\log_a b = n$	then	$b = a^n$

Task

Find the log equivalent of (a) $100 = 10^2$ (b) $\frac{1}{1000} = 10^{-3}$

Here, on the right-hand sides, the base is 10 in each case so:

Your solution

(a) $100 = 10^2$ implies

(b) $\frac{1}{1000} = 10^{-3}$ implies

Answer

(a) $\log_{10} 100 = 2$

(b) $\log_{10} \frac{1}{1000} = -3$

Task

Find the log equivalent of (a) $b = a^n$, (b) $c = a^m$, (c) $bc = a^{n+m}$

(a) Here the base is a so:

Your solution

$b = a^n$ implies $n =$

Answer

$n = \log_a b$

(b) Here the base is a so:

Your solution

$c = a^m$ implies $m =$

Answer

$m = \log_a c$

(c) Here the base is a so:

Your solution

$bc = a^{n+m}$ implies $n + m =$

Answer

$n + m = \log_a(bc)$

From the last Task we have found, using the property of indices, that

$$\log_a(bc) = n + m = \log_a b + \log_a c.$$

We conclude that the index law $a^n a^m = a^{n+m}$ has an equivalent logarithm law

$$\log_a(bc) = \log_a b + \log_a c$$

In words: "The log of a product is the sum of logs."

Indeed this property is one of the major advantages of using logarithms. They transform a **product** of numbers (a relatively difficult operation) to a **sum** of numbers (a relatively easy operation).
Each index law has an equivalent logarithm law, true for any base, listed in the following Key Point:

Key Point 8

The laws of logarithms	The laws of indices
1. $\log_a(AB) = \log_a A + \log_a B$	1. $a^A a^B = a^{A+B}$
2. $\log_a(\frac{A}{B}) = \log_a A - \log_a B$	2. $a^A/a^B = a^{A-B}$
3. $\log_a(A^k) = k \log_a A$	3. $(a^A)^k = a^{kA}$
4. $\log_a(a^A) = A$	4. $a^{\log_a A} = A$
5. $\log_a a = 1$	5. $a^1 = a$
6. $\log_a 1 = 0$	6. $a^0 = 1$

2. Simplifying expressions involving logarithms

To simplify an expression involving logarithms their laws, given in Key Point 8, need to be used.

Example 4

Simplify: $\log_{10} 2 - \log_{10} 4 + \log_{10}(4^2) + \log_{10}(\frac{10}{4})$

Solution

The third term $\log_{10}(4^2)$ simplifies to $2\log_{10} 4$ and the last term

$$\log_{10}(\frac{10}{4}) = \log_{10} 10 - \log_{10} 4 = 1 - \log_{10} 4$$

So $\log_{10} 2 - \log_{10} 4 + \log_{10}(4^2) + \log_{10}(\frac{10}{4}) = \log_{10} 2 - \log_{10} 4 + 2\log_{10} 4 + 1 - \log_{10} 4 = \log_{10} 2 + 1$

Task

Simplify the expression:

$$\log_{10}(\frac{1}{10}) - \log_{10}(\frac{10}{27}) + \log_{10} 1000$$

(a) First simplify $\log_{10}(\frac{1}{10})$:

Your solution

$\log_{10}(\frac{1}{10}) =$

Answer

$\log_{10}(\frac{1}{10}) = \log_{10} 1 - \log_{10} 10 = 0 - 1 = -1$

(b) Now simplify $\log_{10}(\frac{10}{27})$:

Your solution

$\log_{10}(\frac{10}{27}) =$

Answer

$\log_{10}(\frac{10}{27}) = \log_{10} 10 - \log_{10} 27 = 1 - \log_{10} 27$

(c) Now simplify $\log_{10} 1000$:

Your solution

Answer

3

(d) Finally collect all the terms together from (a), (b), (c) and simplify:

Your solution

Answer

$-1 - (1 - \log_{10} 27) + 3 = 1 + \log_{10} 27$

3. Logs to base 10 and natural logs

In practice only two kinds of logarithms are commonly used, those to base 10, written $\log_{10}$ (or just simply $\log$) and those to base e, written $\log_{\mathsf{e}}$ or more usually $\ln$ (called **natural logarithms**). Most scientific calculators will determine the logarithm to base 10 and to base e. For example,

$\log 13 = 1.11394$ (implying $10^{1.11394} = 13$), $\ln 23 = 3.13549$ (implying $\mathsf{e}^{3.13549} = 23$)

Task Use your calculator to determine (a) $\log 10$, (b) $\log 1000000$, (c) $\log 0.1$

Your solution

(a) $\log 10 =$ (b) $\log 1000000 =$ (c) $\log 0.1 =$

Answer

(a) 1, (b) 6, (c) -1.

Each of the above results could be determined directly, without the use of a calculator. For example:
Since $\log_a a = 1$ then $\log 10\ (\equiv \log_{10} 10) = 1$.
Since $\log_a A^k = k \log_a A$ then $\log 1000000 = \log 10^6 = 6 \log 10 = 6$.

Since $\log_a(\dfrac{A}{B}) = \log_a A - \log_a B$ and $\log_a 1 = 0$ and $\log_a a = 0$, then

$$\log 0.1 = \log(\frac{1}{10}) = \log 1 - \log(10) = -1$$

Task Use your calculator to determine
(a) $\ln 29.42$, (b) $\ln \mathrm{e}$, (c) $\ln 0.1$

Your solution

(a) $\ln 29.42 =$ (b) $\ln \mathrm{e} =$ (c) $\ln 0.1 =$

Answer

(a) $\ln 29.42 = 3.38167$, (b) $\ln \mathrm{e} = 1$, (c) $\ln 0.1 = -2.30258$

4. Changing base in logarithms

It is sometimes required to express the logarithm with respect to one base in terms of a logarithm with respect to another base.
Now

$$b = a^n \qquad \text{implies} \qquad \log_a b = n$$

where we have used logs to base a. What happens if, for some reason, we want to use another base, p say? We take logs (to base p) of both sides of $b = a^n$:

$$\log_p(b) = \log_p(a^n) = n \log_p a \qquad \text{(using one of the logarithm laws)}$$

So

$$n = \frac{\log_p(b)}{\log_p(a)} \qquad \text{that is} \qquad \log_a b = \frac{\log_p(b)}{\log_p(a)}$$

This is the rule to be used when converting logarithms from one base to another.

Key Point 9

$$\log_a b = \frac{\log_p b}{\log_p a}$$

For base 10 logs:

$$\log_a b = \frac{\log(b)}{\log(a)}$$

For example,

$$\log_3 7 = \frac{\log 7}{\log 3} = \frac{0.8450980}{0.4771212} = 1.7712437$$

(Check, on your calculator, that $3^{1.7712437} = 7$).
For natural logs:

$$\log_a b = \frac{\ln(b)}{\ln(a)}$$

For example,

$$\log_3 7 = \frac{\ln 7}{\ln 3} = \frac{1.9459101}{1.0986123} = 1.7712437$$

Of course, $\log_3 7$ cannot be determined directly on your calculator since logs to base 3 are not available but it can be found using the above method.

Task

Use your calculator to determine the value of $\log_{21} 7$ using first base 10 then check using base e.

Re-express $\log_{21} 7$ using base 10 then base e:

Your solution

$$\log_{21} 7 = \frac{\log 7}{\log 21} = \qquad \log_{21} 7 = \frac{\ln 7}{\ln 21} =$$

Answer

$$\log_{21} 7 = \frac{\log 7}{\log 21} = 0.6391511 \qquad \log_{21} 7 = \frac{\ln 7}{\ln 21} = 0.6391511$$

Example 5

Simplify the expression $10^{\log x}$.

Solution

Let $y = 10^{\log x}$ then take logs (to base 10) of both sides:

$$\log y = \log(10^{\log x}) = (\log x)\log 10$$

where we have used: $\log A^k = k \log A$. However, since we are using logs to base 10 then $\log 10 = 1$ and so

$$\log y = \log x \qquad \text{implying} \qquad y = x$$

Therefore, finally we conclude that

$$10^{\log x} = x$$

This is an important result true for logarithms of any base. It follows from the basic definition of the logarithm.

Key Point 10

$$a^{\log_a x} = x$$

Raising to the power and taking logs are **inverse** operations.

Exercises

1. Find the values of (a) $\log_2 8$ (b) $\log_{16} 50$ (c) $\ln 28$

2. Simplify

   (a) $\log 1 - 3\log 2 + \log 16$.

   (b) $10\log x - 2\log x^2$.

   (c) $\ln(8x-4) - \ln(4x-2)$.

   (d) $\ln 10 \log 7 - \ln 7$.

Answers

1. (a) 3 (b) 1.41096 (c) 3.3322
2. (a) $\log 2$, (b) $6\log x$ or $\log x^6$, (c) $\ln 2$, (d) 0