The Hyperbolic Functions

6.2

Introduction

The hyperbolic functions $\sinh x$, $\cosh x$, $\tanh x$ etc are certain combinations of the exponential functions e^x and e^{-x}. The notation implies a close relationship between these functions and the trigonometric functions $\sin x$, $\cos x$, $\tan x$ etc. The close relationship is algebraic rather than geometrical. For example, the functions $\cosh x$ and $\sinh x$ satisfy the relation

$$\cosh^2 x - \sinh^2 x \equiv 1$$

which is very similar to the trigonometric identity $\cos^2 x + \sin^2 x \equiv 1$. (In fact every trigonometric identity has an equivalent hyperbolic function identity.)

The hyperbolic functions are not introduced because they are a mathematical nicety. They arise naturally and sufficiently often to warrant sustained study. For example, the shape of a chain hanging under gravity is well described by $\cosh$ and the deformation of uniform beams can be expressed in terms of $\tanh$.

Prerequisites

Before starting this Section you should . . .

- have a good knowledge of the exponential function
- have knowledge of odd and even functions
- have familiarity with the definitions of $\tan$, $\sec$, cosec, $\cot$ and of trigonometric identities

Learning Outcomes

On completion you should be able to . . .

- explain how hyperbolic functions are defined in terms of exponential functions
- obtain and use hyperbolic function identities
- manipulate expressions involving hyperbolic functions

1. Even and odd functions

Constructing even and odd functions

A given function $f(x)$ can always be split into two parts, one of which is even and one of which is odd. To do this write $f(x)$ as $\frac{1}{2}[f(x) + f(x)]$ and then simply add and subtract $\frac{1}{2}f(-x)$ to this to give

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$$

The term $\frac{1}{2}[f(x) + f(-x)]$ is **even** because when x is replaced by $-x$ we have $\frac{1}{2}[f(-x) + f(x)]$ which is the same as the original. However, the term $\frac{1}{2}[f(x) - f(-x)]$ is **odd** since, on replacing x by $-x$ we have $\frac{1}{2}[f(-x) - f(x)] = -\frac{1}{2}[f(x) - f(-x)]$ which is the negative of the original.

Example 2

Separate $x^3 + 2^x$ into odd and even parts.

Solution

$$f(x) = x^3 + 2^x$$

$$f(-x) = (-x)^3 + 2^{-x} = -x^3 + 2^{-x}$$

Even part:

$$\frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}(x^3 + 2^x - x^3 + 2^{-x}) = \frac{1}{2}(2^x + 2^{-x})$$

Odd part:

$$\frac{1}{2}(f(x) - f(-x)) = \frac{1}{2}(x^3 + 2^x + x^3 - 2^{-x}) = \frac{1}{2}(2x^3 + 2^x - 2^{-x})$$

Task

Separate the function $x^2 - 3^x$ into odd and even parts.

First, define $f(x)$ and find $f(-x)$:

Your solution

$f(x) =$ $\qquad\qquad$ $f(-x) =$

Answer

$f(x) = x^2 - 3^x, \quad f(-x) = x^2 - 3^{-x}$

Now construct $\frac{1}{2}[f(x)+f(-x)]$, $\quad \frac{1}{2}[f(x)-f(-x)]$:

Your solution

$\frac{1}{2}[f(x)+f(-x)] =$ $\qquad\qquad$ $\frac{1}{2}[f(x)-f(-x)] =$

Answer

$$\frac{1}{2}[f(x)+f(-x)] = \frac{1}{2}(x^2-3^x+x^2-3^{-x})$$

$$= x^2-\frac{1}{2}(3^x+3^{-x}). \text{ This is the even part of } f(x).$$

$$\frac{1}{2}[f(x)-f(-x)] = \frac{1}{2}(x^2-3^x-x^2+3^{-x})$$

$$= \frac{1}{2}(3^{-x}-3^x). \text{ This is the odd part of } f(x).$$

The odd and even parts of the exponential function

Using the approach outlined above we see that the even part of e^x is

$$\frac{1}{2}(e^x+e^{-x})$$

and the odd part of e^x is

$$\frac{1}{2}(e^x-e^{-x})$$

We give these new functions special names: $\cosh x$ (pronounced 'cosh' x) and $\sinh x$ (pronounced 'shine' x).

Key Point 3

Hyperbolic Functions

$$\cosh x \equiv \frac{1}{2}(e^x+e^{-x})$$

$$\sinh x \equiv \frac{1}{2}(e^x-e^{-x})$$

These two functions, when added and subtracted, give

$$\cosh x+\sinh x \equiv e^x \qquad \text{and} \qquad \cosh x-\sinh x \equiv e^{-x}$$

The graphs of $\cosh x$ and $\sinh x$ are shown in Figure 4.

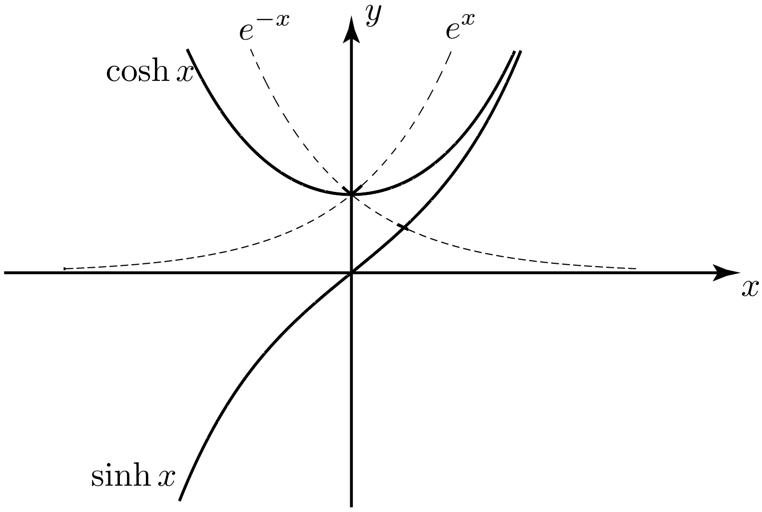

Figure 4: $\sinh x$ and $\cosh x$

Note that $\cosh x > 0$ for all values of x and that $\sinh x$ is zero only when $x = 0$.

2. Hyperbolic identities

The hyperbolic functions $\cosh x,\ \sinh x$ satisfy similar (but not exactly equivalent) identities to those satisfied by $\cos x,\ \sin x$. We note first some basic notation similar to that employed with trigonometric functions:

$$\cosh^n x \quad \text{means} \quad (\cosh x)^n \qquad\qquad \sinh^n x \quad \text{means} \quad (\sinh x)^n \qquad n \neq -1$$

In the special case that $n = -1$ we **do not** use $\cosh^{-1} x$ and $\sinh^{-1} x$ to mean $\dfrac{1}{\cosh x}$ and $\dfrac{1}{\sinh x}$ respectively. The notation $\cosh^{-1} x$ and $\sinh^{-1} x$ is reserved for the **inverse functions** of $\cosh x$ and $\sinh x$ respectively.

Task

Show that $\cosh^2 x - \sinh^2 x \equiv 1$ for all x.

(a) First, express $\cosh^2 x$ in terms of the exponential functions $\mathrm{e}^x,\ \mathrm{e}^{-x}$:

Your solution

$$\cosh^2 x \equiv \left[\frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})\right]^2 \equiv$$

Answer

$$\frac{1}{4}(\mathrm{e}^x + \mathrm{e}^{-x})^2 \equiv \frac{1}{4}[(\mathrm{e}^x)^2 + 2\mathrm{e}^x\mathrm{e}^{-x} + (\mathrm{e}^{-x})^2] \equiv \frac{1}{4}[\mathrm{e}^{2x} + 2\mathrm{e}^{x-x} + \mathrm{e}^{-2x}] \equiv \frac{1}{4}[\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}]$$

(b) Similarly, express $\sinh^2 x$ in terms of e^x and e^{-x}:

Your solution

$$\sinh^2 x \equiv \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \equiv$$

Answer

$\frac{1}{4}(e^x - e^{-x})^2 \equiv \frac{1}{4}[(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2] \equiv \frac{1}{4}[e^{2x} - 2e^{x-x} + e^{-2x}] \equiv \frac{1}{4}[e^{2x} - 2 + e^{-2x}]$

(c) Finally determine $\cosh^2 x - \sinh^2 x$ using the results from (a) and (b):

Your solution

$$\cosh^2 x - \sinh^2 x \equiv$$

Answer

$\cosh^2 x - \sinh^2 x \equiv \frac{1}{4}[e^{2x} + 2 + e^{-2x}] - \frac{1}{4}[e^{2x} - 2 + e^{-2x}] \equiv 1$

As an alternative to the calculation in this Task we could, instead, use the relations

$$e^x \equiv \cosh x + \sinh x \qquad\qquad e^{-x} \equiv \cosh x - \sinh x$$

and remembering the algebraic identity $(a+b)(a-b) \equiv a^2 - b^2$, we see that

$$(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv e^x e^{-x} \equiv 1 \qquad \text{that is} \qquad \cosh^2 x - \sinh^2 x \equiv 1$$

Key Point 4

The fundamental identity relating hyperbolic functions is:

$$\cosh^2 x - \sinh^2 x \equiv 1$$

This is the hyperbolic function equivalent of the trigonometric identity: $\cos^2 x + \sin^2 x \equiv 1$

Task Show that $\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y$.

First, express $\cosh x \cosh y$ in terms of exponentials:

Your solution

$$\cosh x \cosh y \equiv \left(\frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{y}+\mathrm{e}^{-y}}{2}\right) \equiv$$

Answer

$$\left(\frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{y}+\mathrm{e}^{-y}}{2}\right) \equiv \frac{1}{4}[\mathrm{e}^{x}\mathrm{e}^{y}+\mathrm{e}^{-x}\mathrm{e}^{y}+\mathrm{e}^{x}\mathrm{e}^{-y}+\mathrm{e}^{-x}\mathrm{e}^{-y}] \equiv \frac{1}{4}(\mathrm{e}^{x+y}+\mathrm{e}^{-x+y}+\mathrm{e}^{x-y}+\mathrm{e}^{-x-y})$$

Now express $\sinh x \sinh y$ in terms of exponentials:

Your solution

$$\left(\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{y}-\mathrm{e}^{-y}}{2}\right) \equiv$$

Answer

$$\left(\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{y}-\mathrm{e}^{-y}}{2}\right) \equiv \frac{1}{4}(\mathrm{e}^{x+y}-\mathrm{e}^{-x+y}-\mathrm{e}^{x-y}+\mathrm{e}^{-x-y})$$

Now express $\cosh x \cosh y + \sinh x \sinh y$ in terms of a hyperbolic function:

Your solution

$$\cosh x \cosh y + \sinh x \sinh y =$$

Answer

$\cosh x \cosh y + \sinh x \sinh y \equiv \frac{1}{2}(\mathrm{e}^{x+y}+\mathrm{e}^{-(x+y)})$ which we recognise as $\cosh(x+y)$

Other hyperbolic function identities can be found in a similar way. The most commonly used are listed in the following Key Point.

Key Point 5

Hyperbolic Identities

- $\cosh^2 - \sinh^2 \equiv 1$
- $\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y$
- $\sinh(x+y) \equiv \sinh x \cosh y + \cosh x \sinh y$
- $\sinh 2x \equiv 2\sinh x \cosh y$
- $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$ or $\cosh 2x \equiv 2\cosh^2 - 1$ or $\cosh 2x \equiv 1 + 2\sinh^2 x$

3. Related hyperbolic functions

Given the trigonometric functions $\cos x,\ \sin x$ related functions can be defined; $\tan x,\ \sec x,\ \text{cosec } x$ through the relations:

$$\tan x \equiv \frac{\sin x}{\cos x} \qquad \sec x \equiv \frac{1}{\cos x} \qquad \text{cosec } x \equiv \frac{1}{\sin x} \qquad \cot x \equiv \frac{\cos x}{\sin x}$$

In an analogous way, given $\cosh x$ and $\sinh x$ we can introduce hyperbolic functions $\tanh x,\ \sec\text{h } x$, $\text{cosech } x$ and $\coth x$. These functions are defined in the following Key Point:

Key Point 6

Further Hyperbolic Functions

$$\begin{aligned} \tanh x &\equiv \frac{\sinh x}{\cosh x} \\ \text{sech } x &\equiv \frac{1}{\cosh x} \\ \text{cosech } x &\equiv \frac{1}{\sinh x} \\ \coth x &\equiv \frac{\cosh x}{\sinh x} \end{aligned}$$

Show that $1 - \tanh^2 x \equiv \text{sech}^2 x$

Use the identity $\cosh^2 x - \sinh^2 x \equiv 1$:

Your solution

Answer

Dividing both sides by $\cosh^2 x$ gives

$$1 - \frac{\sinh^2 x}{\cosh^2 x} \equiv \frac{1}{\cosh^2 x} \qquad \text{implying (see Key Point 6)} \qquad 1 - \tanh^2 x \equiv \text{sech}^2 x$$

Exercises

1. Express
   (a) $2\sinh x + 3\cosh x$ in terms of e^x and e^{-x}.
   (b) $2\sinh 4x - 7\cosh 4x$ in terms of e^{4x} and e^{-4x}.
2. Express
   (a) $2\mathrm{e}^x - \mathrm{e}^{-x}$ in terms of $\sinh x$ and $\cosh x$.
   (b) $\dfrac{7\mathrm{e}^x}{(\mathrm{e}^x - \mathrm{e}^{-x})}$ in terms of $\sinh x$ and $\cosh x$, and then in terms of $\coth x$.
   (c) $4\mathrm{e}^{-3x} - 3\mathrm{e}^{3x}$ in terms of $\sinh 3x$ and $\cosh 3x$.
3. Using only the $\cosh$ and $\sinh$ keys on your calculator (or e^x key) find the values of
   (a) $\tanh 0.35$, (b) $\text{cosech } 2$, (c) $\text{sech } 0.6$.

Answers

1. (a) $\frac{5}{2}\mathrm{e}^x - \frac{1}{2}\mathrm{e}^{-x}$ (b) $-\frac{5}{2}\mathrm{e}^{4x} - \frac{9}{2}\mathrm{e}^{-4x}$

2. (a) $\cosh x + 3\sinh x$, (b) $\dfrac{7(\cosh x + \sinh x)}{2\sinh x}$, $\dfrac{7}{2}(\coth x + 1)$ (c) $\cosh 3x - 7\sinh 3x$

3. (a) 0.3364, (b) 0.2757 (c) 0.8436