Quadratic Functions and Modelling

5.2

Introduction

This Section describes forms of equations for quadratic functions (also called parabolas), ways in which quadratic functions can be used to model motion involving projectiles, and certain kinds of problem involving a single maximum or minimum.

Prerequisites

Before starting this Section you should ...

- be competent at algebraic manipulation
- be familiar with quadratic functions

Learning Outcomes

On completion you should be able to ...

- use quadratic functions to model motion under constant acceleration
- express the equation of a parabola in a general form

1. Quadratic functions

Quadratic functions and parabolas

Graphs of y against x resulting from quadratic functions (HELM 2.8, Table 1) are called **parabolas**. These take the general form: $y = ax^2 + bx + c$. The coefficients a, b and c influence the shape, form and position of the graph of the associated parabola. They are the **parameters** of the parabola. In particular the magnitude of a determines how wide the parabola opens (large a implies a narrow parabola, small a implies a wide parabola) and the sign of a determines whether the parabola has a lowest point (minimum) or highest point (maximum). Negative a implies a parabola with a highest point. The most useful form of equation for determing the graphical appearance of a parabola is $y - C = A(x - B)^2$). To see the relation between this form and the general form simply expand:

$$y = Ax^2 - 2ABx + AB^2 + C$$

so, comparing with $y = ax^2 + bx + c$ we have:

$$a \equiv A, \quad b \equiv -2AB \quad c \equiv AB^2 + C$$

We deduce that the relation between the two sets of constants A, B, C and a, b, c is:

$$A = a \qquad B = -\frac{b}{2a} \quad \text{and} \quad C = c - \frac{b^2}{4a}$$

This new form for the parabola enables the coordinates of the highest or lowest point, known as the **vertex** to be written down immediately. The coordinates of the vertex are given by (B, C). Changing the value of B shifts the vertex, and hence the whole parabola, up or down. Changing the value of C shifts the vertex, and hence the whole parabola, to left or right.

Task

Assume the variation of an object's location with time is represented by a quadratic function:-

$$s = \frac{t^2}{9} \qquad (0 \leq t \leq 30)$$

Compare this function with the general form $y - C = A(x - B)^2$.

(a) What variables correspond to y and x in this case?

(b) What are the values of C, A and B?

Your solution

Answer

(a) s corresponds to y, and t corresponds to x (b) $C = 0, A = \frac{1}{9}$ and $B = 0$

2. Modelling with parabolas

The function

$$s = \frac{t^2}{9} \qquad (0 \le t \le 30)$$

is part of a parabola starting at the origin ($s = 0$ and $t = 0$) and rising to $s = 100$ at the end of its range of validity. s represents the distance of the object from the origin - N.B. Do not confuse this s with the symbol for seconds. 'Negative' time corresponds to time before the motion of the object is being considered. What would this parabolic function have predicted if it were valid up to 30 s before the 'zero' time? The answer to this can be deduced from the left-hand part of the graph of the function shown as a dashed curve, for in Figure 4, i.e. the part corresponding to $-30 \le t \le 0$.

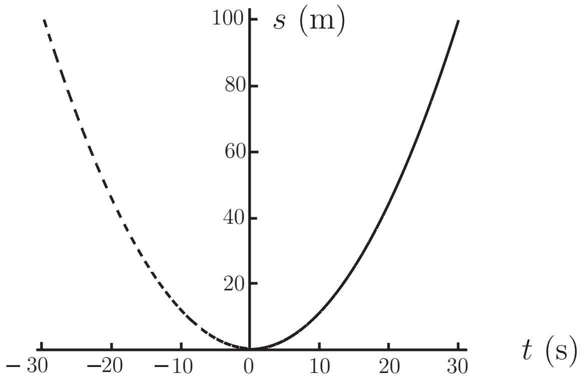


Figure 4: Graph of $s = \frac{t^2}{9}$ for $-30 \le t \le 30$

The parabolic form predicts that at $t = -30$, the object was 100 m away and for $(-30 \le t \le 0)$ it was moving towards the point at which the original timing started. The rate of change of position, or instantaneous velocity, is given by the gradient of the position-time graph. Since the gradient of the parabola for s is steeper near $t = -30$ than near $t = 0$, the chosen function for s and new range of validity suggests that the object was moving quickly at the start of the motion, slows down on approaching the initial starting point, and then moves away again accelerating as it does so. Note that the velocity (i.e. the gradient) for $(-30 \le t \le 0)$ is negative while for $(0 \le t \le 30)$ it is positive. This is consistent with the change in direction at $t = 0$.

We will consider falling objects again and return to the context of the thriller film and the villain on a cliff-tip dislodging a rock. Suppose that, as film director, you are considering a variation of the plot whereby, instead of the ground, the rock hits the roof of a vehicle carrying the hero and heroine. This means that you might be interested in the position as well as the velocity of the rock at any time. We can start from the linear function relating velocity and time for the dislodged rock,

$$v = 9.8t \qquad (0 \le t \le T)$$

where T represents the time at which the rock hits the roof of the vehicle. The precise value of T will depend upon the height of the vehicle. If s is measured *from the cliff-top* and timing starts with release of the rock, so that $s = 0$ when $t = 0$, the resulting function is

$$s = 4.9t^2 \qquad (0 \le t \le T)$$

(Note that $s = 4.9t^2$ is a particular case of a standard model for falling objects: $s = \frac{1}{2}gt^2$.)

Task

This Task refers to the model discussed above.

(a) What kind of function is $s = 4.9t^2$?

Your solution

Answer

Quadratic, or parabolic

(b) If the vehicle roof is 2 m above the ground and the cliff-top is 35 m above the ground, calculate a value for T, the time when a rock falling from the cliff-top hits the car roof:

Your solution

Answer

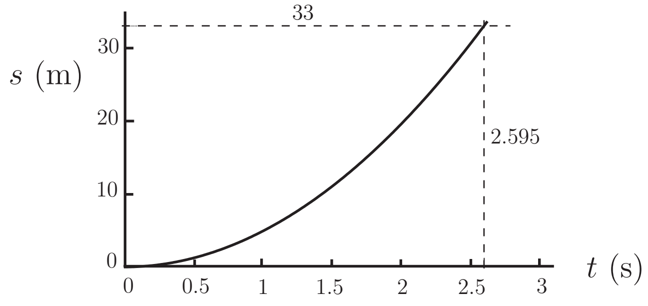
$t = T$ when $s = 35 - 2 = 33$

$$\Rightarrow \quad 33 = 4.9T^2 \qquad \text{so} \quad T = \sqrt{\frac{33}{4.9}} = 2.5951 \approx 2.6 \quad \text{(only positive } T \text{ makes sense)}$$

(c) Given this value for T sketch the function:

Your solution

Answer



In this modelling context, negative time would correspond to time before the villain dislodges the rock. It seems likely that the rock was stationary before this instant. The parabolic function would not be appropriate for $t \leq 0$ since it would predict that the rock was moving. An appropriate function would have two parts to its domain:

For $t \leq 0, s$ would be constant $(= 0)$ and for $0 \leq t \leq T,\ s = 4.9t^2$.

The corresponding graph would also have two parts:

A flat line along the $s = 0$ axis for $t \leq 0$ and part of a parabola for $0 < t \leq T$.

A different form of quadratic function for position is appropriate if position is measured **upwards** as height (h) above the ground below the cliff-top. This is given as

$h = 35 - 4.9t^2 \qquad (0 \leq t \leq 2.6)$

Note that once $t = 2.6$ then $h = 0$ and the rock cannot fall any further. When position is measured upwards, velocities and accelerations, which are downwards for falling objects, will be negative.

Task

This Task refers to the model discussed above.

By comparing $h = 35 - 4.9t^2$ with $y = ax^2 + bx + c$, deduce values for a, b and c and determine whether the parabola corresponding to this function has a highest or lowest point:

Your solution

Answer

Here h corresponds to y and t to x in the general form. The coefficient corresponding to a is $-4.9 \times b = 0$ and $c = 35$. The value of a is negative so the parabola opens downwards.

(b) Write down an appropriate function for the variation of h with t if height is measured upwards from the top of a 2 m high vehicle:

Your solution

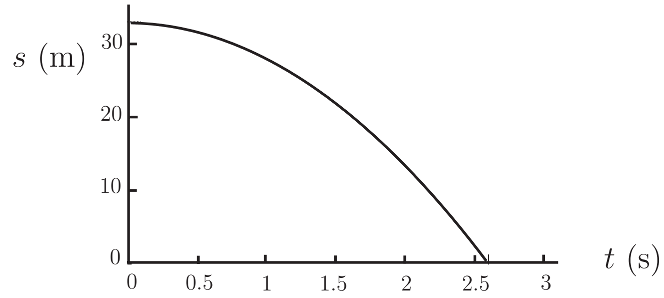
Answer

$h = 33 - 4.9t^2 \qquad (0 \leq t \leq 2.5951) = 2.60$ to 2 d.p.

(c) Sketch this function:

Your solution

Answer



Consider the situation in which position is measured **downwards** from the cliff-top again but the villain is lying down on the cliff-top and throws the rock **upwards** with speed 5 m s^{-1}. The distance it would travel in time t seconds if gravity were not acting would be $-5t$ metres (distance is speed multiplied by time but in the negative s direction in this case). To obtain the resulting distance in the presence of gravity we add this to the distance function $s = 4.9t^2$ that applies when the rock is simply dropped. The appropriate quadratic function for s is now

$$s = 4.9t^2 - 5t \qquad (0 \leq t \leq T)$$

The nature of this quadratic function means that for any given value of s there are **two** possible values of t. If we write the function in a slightly different way, taking out a common factor of t,

$$s = t(4.9t - 5) \qquad (0 \leq t \leq T)$$

it is possible to see that $s = 0$ at two different times. These are when $t = 0$ and when $4.9t - 5 = 0$. The first possibility is consistent with the initial position of the rock. The second possibility gives $t = \dfrac{5}{4.9}$ which is a little more than 1. The rock will be at the cliff-top level at two different times. It is there at the instant when it is thrown. It rises until its speed is zero and then descends, passing cliff-top level again on its way to impact with the ground below or with the vehicle roof. Since the initial motion of the rock is upwards and position is defined as positive downwards, the initial part of the rock's path corresponds to negative s. The parabola associated with the appropriate function crosses the $s = 0$ axis twice and has a vertex at which s is negative. A sketch of s against t for this case is shown in Figure 5.

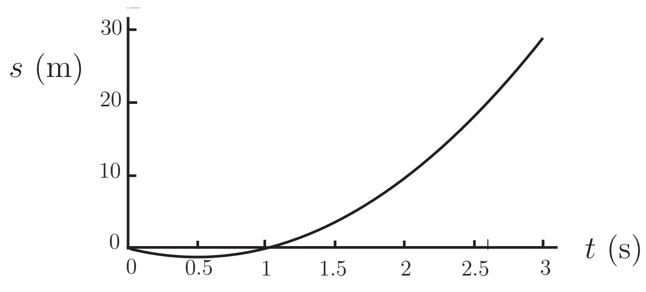


Figure 5: Graph of rock's position (measured downwards) when rock is thrown upwards

Task

For the above modelling of falling rocks, calculate how high the rock rises after being thrown upwards at 5 m s^{-1}. (Hint: use the previously determined value of the time when the rock reaches its highest point.)

Your solution

Answer

The value of t at which the rock's velocity is zero was worked out as $t = \dfrac{5}{9.8}$. This value can be used in the function for s to give

$$s = \frac{5}{9.8}(4.9 \times \frac{5}{9.8} - 5) = -\frac{2.5}{19.6} = -1.2755$$

So the rock rises to a little less than 1.28 m above the cliff-top.

Note that the form of the parabola makes it inevitable that, as long as it is plotted over a sufficiently wide range, and apart from its vertex, there will **always be two values** on the curve for each value of one of the variables. Which of these values makes sense in a mathematical model will depend on the modelling context. In each of the contexts mentioned so far in this Section each context has determined the part of the parabola that is of interest.
Note also that there is a connection between the vertex on a parabola and the point where the gradient of that parabola is zero. In fact these points are the same!

3. Parabolas and optimisation

Because the vertex may represent a highest or lowest point, a quadratic function may be the appropriate type of function to choose in a modelling problem where a maximum or a minimum is involved (optimisation problems for example). Consider the problem of working out the selling price for the product of a cottage industry that would maximise the profit, given certain details of costs and assumptions about market behaviour. A possible function relating profit ($\pounds M$) to selling price ($\pounds P$), is

$$M = -10P^2 + 320P - 2420 \qquad (12 \leq P \leq 20).$$

Note that this is a quadratic function. By comparing this function with the form $y = ax^2 + bx + c$ it is possible to decide whether the corresponding parabola that would result from graphing M against P, would open upwards or downwards. Here M corresponds to y and P to x. The coefficient corresponding to a in the general form is -10. This is negative, so the resulting parabola will open downwards. In other words it will have a **highest point** or **maximum** for some value of P. This is comforting in the context of an optimisation problem! We can go further in specifying the resulting parabola by reference to the other general form: $y - C = A(x - B)^2$. If we multiply out the bracket on the right hand side we get (as seen at the beginning of HELM 5.2)

$$y - C = Ax^2 - 2ABx + AB^2$$

or

$$y = Ax^2 - 2ABx + AB^2 + C.$$

Comparing this general form with the function relating profit and price for the cottage industry:

$$y = Ax^2 - 2ABx + AB^2 + C$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$M = -10P^2 + \quad 320P \quad - 2420$$

Using the equivalances suggested by the arrows, we see that

$$A = -10,$$

$$2AB = -320$$

$$AB^2 + C = -2420.$$

These are three equations for three unknowns. Putting $A = -10$ in the second equation gives $B = 16$. Putting $A = -10$ and $B = 16$ in the third equation gives

$$-2560 + C = -2420,$$

and so

$$C = 140.$$

This means that the equation for M may also be written in the form

$$M - 140 = -10(P - 16)^2,$$

corresponding to the general form $y - C = A(x - B)^2$. In the general form, C corresponds to the value of y at the vertex of the parabola. Since y in the general form corresponds to M in the current modelling context, we deduce that $M = 140$ at the highest point on the parabola. B represents the value of x at the lowest or highest point of the general parabola. Here x corresponds to P, so we deduce thet $P = 16$ at the vertex of the parabola corresponding to the function relating profit and price. These deductions mean that a maximum profit of £140 is obtained when the selling price is £16.

4. Finding the equation of a parabola

Consider a parabola that has its vertex at $s = 50$ when $t = 0$ and rises to $s = 100$ when $t = 30$. In coordinate terms, we need the equation of a parabola that has its lowest point or vertex at (0, 50) and passes through (30, 100). The general form

$$y - C = A(x - B)^2$$

is useful here.
In this case y corresponds to s and x to t. So the equation relating s and t is

$$s - C = A(t - B)^2$$

According to the general form, the coordinates of the vertex are (B, C). We know that the coordinates of the vertex are (0, 50). So we can deduce that $B = 0$ and $C = 50$. It remains to find A. The fact that the parabola must pass through (30, 100) may be used for this purpose. These values together with those for B and C may be substituted in the general equation:

$$100 - 50 = A(30 - 0)^2$$

so $50 = 900A$ or $A = \dfrac{1}{18}$ and the function we want is

$$s = 50 + \frac{1}{18}t^2 \qquad (0 \leq t \leq 30)$$

Task

Find the equation of a parabola with vertex at $(0, 2)$ and passing through the point $(4, 4)$.

Your solution

Answer

Using the general form, with $B = 0$ and $C = 2$,

$$y - 2 = A(x - 0)^2 \text{ or } y - 2 = Ax^2$$

Then using the point $(4, 4)$

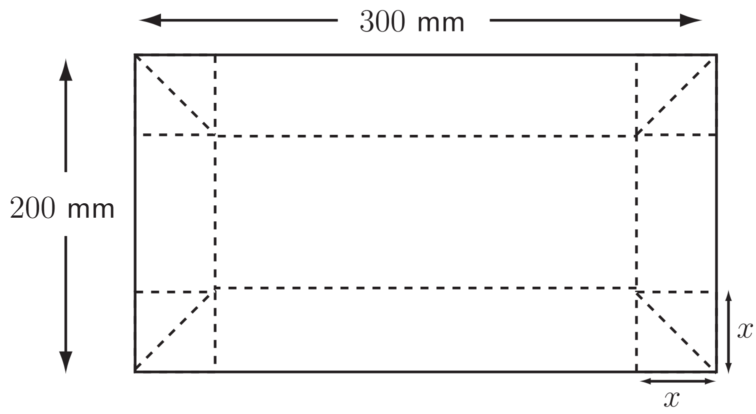
$$4 - 2 = 16A \qquad \text{so} \qquad A = \frac{2}{16} = \frac{1}{8}$$

and the required equation is

$$y = 2 + \frac{1}{8}x^2$$

Exercise

An open-topped carton is constructed from a 200 mm × 300 mm sheet of cardboard, using simple folds as shown in the diagram.



Cardboard folds to make an open-topped carton

(a) Show that the volume of the carton (in cm^3) is

$$V = \frac{x(300-2x)(200-2x)}{1000}$$

so $$V = \frac{x^3}{250} - x^2 + 60x \qquad \ldots(*)$$

(b) Sketch Equation (1) as V vs x and hence estimate the maximum volume of carton that may be obtained by folding the cardboard sheet.

(c) A carton with a volume of 1000 cm^3 is to be made from the cardboard sheet.

(i) Show that one solution is to use a height $x = 50$ mm.

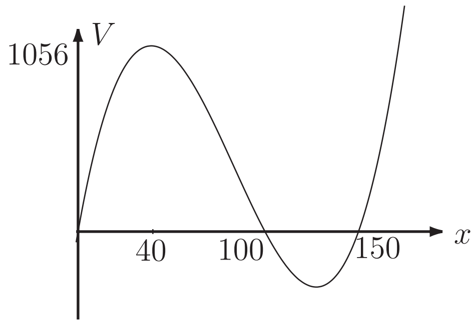
(ii) By factorisation of Equation (*) for $V = 1000$ cm^3, find a second solution for x which would give the same carton volume.

(iii) Why does the third root have no physical meaning?

Answer

(a) $V = \dfrac{x(300-2x)(200-2x)}{1000} = \dfrac{x^3}{250} - x^2 + 60x$ (cm^3)

(b)



$V_{\text{max}} \approx 1056$ cm^3 when $x \approx 39.2$

(c) (i) $x = 50$ mm $\Rightarrow$ $V = 1000$ cm^3 as required.

(ii) $\dfrac{x^3 - 250x^2 + 15000x}{250} - 1000 = 0$ factorises to

$(x-50)(x^2 - 200x + 5000) = 0$

so $x = 50$ or $x = 100 \pm 10\sqrt{50} \approx 29.3$ or 170.7. The second root is 29.3.

(iii) The third root 170.7 is impossible as $200 - 2x$ must be a positive distance.