Trigonometric Identities

4.3

Introduction

A trigonometric identity is a relation between trigonometric expressions which is true for all values of the variables (usually angles). There are a very large number of such identities. In this Section we discuss only the most important and widely used. Any engineer using trigonometry in an application is likely to encounter some of these identities.

Prerequisites

Before starting this Section you should ...

- have a basic knowledge of the geometry of triangles

Learning Outcomes

On completion you should be able to ...

- use the main trigonometric identities
- use trigonometric identities to combine trigonometric functions

1. Trigonometric identities

An identity is a relation which is always true. To emphasise this the symbol '$\equiv$' is often used rather than '$=$'. For example, $(x+1)^2 \equiv x^2+2x+1$ (always true) but $(x+1)^2 = 0$ (only true for $x = -1$).

Task

(a) Using the exact values, evaluate $\sin^2\theta + \cos^2\theta$ for (i) $\theta = 30°$ (ii) $\theta = 45°$

[Note that $\sin^2\theta$ means $(\sin\theta)^2$, $\cos^2\theta$ means $(\cos\theta)^2$]

(b) Choose a non-integer value for θ and use a calculator to evaluate $\sin^2\theta + \cos^2\theta$.

Your solution

Answer

(a) (i) $\sin^2 30° + \cos^2 30° = \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{1}{4} + \dfrac{3}{4} = 1$

(ii) $\sin^2 45° + \cos^2 45° = \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2} + \dfrac{1}{2} = 1$

(b) The answer should be 1 whatever value you choose.

Key Point 12

For any value of θ

$$\sin^2\theta + \cos^2\theta \equiv 1 \qquad (5)$$

One way of proving the result in Key Point 12 is to use the definitions of $\sin\theta$ and $\cos\theta$ obtained from the circle of unit radius. Refer back to Figure 22 on page 23.
Recall that $\cos\theta = OQ$, $\sin\theta = OR = QP$. By Pythagoras' theorem

$$(OQ)^2 + (QP)^2 = (OP)^2 = 1$$

hence $\cos^2\theta + \sin^2\theta = 1$.
We have demonstrated the result (5) using an angle θ in the first quadrant but the result is true for any θ i.e. it is indeed an identity.

Task

By dividing the identity $\sin^2\theta + \cos^2\theta \equiv 1$ by (a) $\sin^2\theta$ (b) $\cos^2\theta$ obtain two further identities.

[Hint: Recall the definitions of $\text{cosec}\,\theta,\ \sec\theta,\ \cot\theta$.]

Your solution

Answer

(a) $\dfrac{\sin^2\theta}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}$ (b) $\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$

$\boxed{1 + \cot^2\theta \equiv \text{cosec}^2\theta}$ $\boxed{\tan^2\theta + 1 \equiv \sec^2\theta}$

Key Point 13 introduces two further important identities.

Key Point 13

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B \qquad (6)$$

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B \qquad (7)$$

Note carefully the addition sign in (6) but the subtraction sign in (7).

Further identities can readily be obtained from (6) and (7).
Dividing (6) by (7) we obtain

$$\tan(A+B) \equiv \frac{\sin(A+B)}{\cos(A+B)} \equiv \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

Dividing every term by $\cos A\ \cos B$ we obtain

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Replacing B by $-B$ in (6) and (7) and remembering that $\cos(-B) \equiv \cos B,\ \sin(-B) \equiv -\sin B$ we find

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

Task

Using the identities $\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$ and

$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$ obtain an expansion for $\tan(A-B)$:

Your solution

Answer

$$\tan(A-B) \equiv \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}.$$

Dividing every term by $\cos A \ \cos B$ gives

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

The following identities are derived from those in Key Point 13.

Key Point 14

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad (8)$$

$$\sin(A-B) \equiv \sin A \cos B - \cos A \sin B \qquad (9)$$

$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B \qquad (10)$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B} \qquad (11)$$

Engineering Example 5

Amplitude modulation

Introduction

Amplitude Modulation (the AM in AM radio) is a method of sending electromagnetic signals of a certain frequency (signal frequency) at another frequency (carrier frequency) which may be better for transmission. Modulation can be represented by the multiplication of the carrier and modulating signals. To **demodulate** the signal the carrier frequency must be removed from the modulated signal.

Problem in words

(a) A single frequency of 200 Hz (message signal) is amplitude modulated with a carrier frequency of 2 MHz. Show that the modulated signal can be represented by the sum of two frequencies at $2 \times 10^6 \pm 200$ Hz

(b) Show that the modulated signal can be demodulated by using a locally generated carrier and applying a low-pass filter.

Mathematical statement of problem

(a) Express the message signal as $m = a\cos(\omega_m t)$ and the carrier as $c = b\cos(u_c t)$.

Assume that the modulation gives the product $mc = ab\cos(u_c t)\cos(\omega_m t)$.

Use trigonometric identities to show that

$$mc = ab\cos(\omega_c t)\cos(u_m t) = k_1\cos((\omega_c - u_m)t) + k_2\cos((\omega_c + u_m)t)$$

where k_1 and k_2 are constants.

Then substitute $\omega_c = 2 \times 10^6 \times 2\pi$ and $\omega_m = 200 \times 2\pi$ to calculate the two resulting frequencies.

(b) Use trigonometric identities to show that multiplying the modulated signal by $b\cos(u_c t)$ results in the lowest frequency component of the output having a frequency equal to the original message signal.

Mathematical analysis

(a) The message signal has a frequency of $f_m = 200$ Hz so $\omega_m = 2\pi f_c = 2\pi \times 200 = 400\pi$ radians per second.

The carrier signal has a frequency of $f_c = 2 \times 10^6$ Hz.

Hence $\omega_c = 2\pi f_c = 2\pi \times 2 \times 10^6 = 4 \times 10^6\pi$ radians per second.

So $mc = ab\cos(4 \times 10^6\pi t)\cos(400\pi t)$.

Key Point 13 includes the identity:

$$\cos(A + B) + \cos(A - B) \equiv 2\cos(A)\cos(B)$$

Rearranging gives the identity:

$$\cos(A)\cos(B) \equiv \tfrac{1}{2}(\cos(A+B)+\cos(A-B)) \qquad (1)$$

Using (1) with $A = 4\times 10^6\pi t$ and $B = 400\pi t$ gives

$$\begin{aligned} mc &= ab(\cos(4\times 10^6\pi t)\cos(400\pi t) \\ &= ab(\cos(4\times 10^6\pi t + 400\pi t) + \cos(4\times 10^6\pi t - 400\pi t)) \\ &= ab(\cos(4000400\pi t) + \cos(3999600\pi t)) \end{aligned}$$

So the modulated signal is the **sum of two waves** with angular frequency of 4000400π and 3999600π radians per second corresponding to frequencies of $4000400\pi/(2\pi)$ and $39996000\pi/(2\pi)$, that is 2000200 Hz and 1999800 Hz i.e. $2\times 10^6 \pm 200$ Hz.

(b) Taking identity (1) and multiplying through by $\cos(A)$ gives

$$\cos(A)\cos(A)\cos(B) \equiv \tfrac{1}{2}\cos(A)(\cos(A+B)+\cos(A-B))$$

so

$$\cos(A)\cos(A)\cos(B) \equiv \tfrac{1}{2}(\cos(A)\cos(A+B)+\cos(A)\cos(A-B)) \qquad (2)$$

Identity (1) can be applied to both expressions in the right-hand side of (2). In the first expression, using $A+B$ instead of 'B', gives

$$\cos(A)\cos(A+B) \equiv \frac{1}{2}(\cos(A+A+B)+\cos(A-A-B)) \equiv \frac{1}{2}(\cos(2A+B)+\cos(B))$$

where we have used $\cos(-B)\equiv\cos(B)$.

Similarly, in the second expression, using $A-B$ instead of 'B', gives

$$\cos(A)\cos(A-B) \equiv \tfrac{1}{2}(\cos(2A-B)+\cos(B))$$

Together these give:

$$\begin{aligned} \cos(A)\cos(A)\cos(B) &\equiv \frac{1}{2}(\cos(2A+B)+\cos(B)+\cos(2A-B)+\cos(B)) \\ &\equiv \cos(B)+\frac{1}{2}(\cos(2A+B)+\cos(2A-B)) \end{aligned}$$

With $A = 4\times 10^6\pi t$ and $B = 400\pi t$ and substituting for the given frequencies, the modulated signal multiplied by the original carrier signal gives

$$\begin{aligned} &ab^2\cos(4\times 10^6\pi t)\cos(4\times 10^6\pi t)\cos(400\pi t) = \\ &\qquad ab^2\cos(2\pi\times 200t) + \tfrac{1}{2}ab^2(\cos(2\times 4\times 10^6\pi t + 400\pi t) + \cos(2\times 4\times 10^6\pi t - 400\pi t)) \end{aligned}$$

The last two terms have frequencies of $4\times 10^6 \pm 200$ Hz which are sufficiently high that a low-pass filter would remove them and leave only the term

$$ab^2\cos(2\pi\times 200t)$$

which is the original message signal multiplied by a constant term.

Interpretation

Amplitude modulation of a single frequency message signal (f_m) with a single frequency carrier signal (f_c) can be shown to be equal to the sum of two cosines with frequencies $f_c \pm f_m$. Multiplying the modulated signal by a locally generated carrier signal and applying a low-pass filter can reproduce the frequency, f_m, of the message signal.

This is known as **double side band amplitude modulation**.

Example 2

Obtain expressions for $\cos\theta$ in terms of the sine function and for $\sin\theta$ in terms of the cosine function.

Solution

Using (9) with $A = \theta,\ B = \dfrac{\pi}{2}$ we obtain

$$\cos\left(\theta - \frac{\pi}{2}\right) \equiv \cos\theta\ \cos\left(\frac{\pi}{2}\right) + \sin\theta\ \sin\left(\frac{\pi}{2}\right) \equiv \cos\theta\ (0) + \sin\theta\ (1)$$

i.e. $\sin\theta \equiv \cos\left(\theta - \dfrac{\pi}{2}\right) \equiv \cos\left(\dfrac{\pi}{2} - \theta\right)$

This result explains why the graph of $\sin\theta$ has exactly the same shape as the graph of $\cos\theta$ but it is shifted to the right by $\dfrac{\pi}{2}$. (See Figure 29 on page 28). A similar calculation using (6) yields the result

$$\cos\theta \equiv \sin\left(\theta + \frac{\pi}{2}\right).$$

Double angle formulae

If we put $B = A$ in the identity given in (6) we obtain Key Point 15:

Key Point 15

$$\sin 2A \equiv \sin A\cos A + \cos A\sin A \quad \text{so} \quad \sin 2A \equiv 2\sin A\cos A \qquad (12)$$

Task

Substitute $B = A$ in identity (7) in Key Point 13 on page 38 to obtain an identity for $\cos 2A$. Using $\sin^2 A + \cos^2 A \equiv 1$ obtain two alternative forms of the identity.

Your solution

Answer

Using (7) with $B \equiv A$

$$\begin{aligned} \cos(2A) &\equiv (\cos A)(\cos A) - (\sin A)(\sin A) \\ \therefore \quad \cos(2A) &\equiv \cos^2 A - \sin^2 A \qquad (13) \end{aligned}$$

Substituting for $\sin^2 A$ in (13) we obtain

$$\begin{aligned} \cos 2A &\equiv \cos^2 A - (1 - \cos^2 A) \\ &\equiv 2\cos^2 A - 1 \qquad (14) \end{aligned}$$

Alternatively substituting for $\cos^2 A$ in (13)

$$\begin{aligned} \cos 2A &\equiv (1 - \sin^2 A) - \sin^2 A \\ \cos 2A &\equiv 1 - 2\sin^2 A \qquad (15) \end{aligned}$$

Task

Use (14) and (15) to obtain, respectively, $\cos^2 A$ and $\sin^2 A$ in terms of $\cos 2A$.

Your solution

Answer

From (14) $\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$. From (15) $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$.

Task Use (12) and (13) to obtain an identity for $\tan 2A$ in terms of $\tan A$.

Your solution

Answer

$$\tan 2A \equiv \frac{\sin 2A}{\cos 2A} \equiv \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

Dividing numerator and denominator by $\cos^2 A$ we obtain

$$\tan 2A \equiv \frac{2\dfrac{\sin A}{\cos A}}{1 - \dfrac{\sin^2 A}{\cos^2 A}} \equiv \frac{2\tan A}{1 - \tan^2 A} \qquad (16)$$

Half-angle formulae

If we replace A by $\dfrac{A}{2}$ and, consequently $2A$ by A, in (12) we obtain

$$\sin A \equiv 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) \qquad (17)$$

Similarly from (13)

$$\cos A \equiv 2\cos^2\left(\frac{A}{2}\right) - 1. \qquad (18)$$

These are examples of **half-angle formulae**. We can obtain a half-angle formula for $\tan A$ using (16). Replacing A by $\dfrac{A}{2}$ and $2A$ by A in (16) we obtain

$$\tan A \equiv \frac{2\tan\left(\dfrac{A}{2}\right)}{1 - \tan^2\left(\dfrac{A}{2}\right)} \qquad (19)$$

Other formulae, useful for integration when trigonometric functions are present, can be obtained using (17), (18) and (19) shown in the Key Point 16.

Key Point 16

If $t = \tan\left(\dfrac{A}{2}\right)$ then

$$\sin A = \frac{2t}{1+t^2} \tag{20}$$

$$\cos A = \frac{1-t^2}{1+t^2} \tag{21}$$

$$\tan A = \frac{2t}{1-t^2} \tag{22}$$

Sum of two sines and sum of two cosines

Finally, in this Section, we obtain results that are widely used in areas of science and engineering such as vibration theory, wave theory and electric circuit theory.
We return to the identities (6) and (9)

$$\begin{aligned} \sin(A+B) &\equiv \sin A\cos B + \cos A\sin B \\ \sin(A-B) &\equiv \sin A\cos B - \cos A\sin B \end{aligned}$$

Adding these identities gives

$$\sin(A+B) + \sin(A-B) \equiv 2\sin A\cos B \tag{23}$$

Subtracting the identities produces

$$\sin(A+B) - \sin(A-B) \equiv 2\cos A\sin B \tag{24}$$

It is now convenient to let $C = A+B$ and $D = A-B$ so that

$$A = \frac{C+D}{2} \quad \text{and} \quad B = \frac{C-D}{2}$$

Hence (23) becomes

$$\sin C + \sin D \equiv 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \tag{25}$$

Similarly (24) becomes

$$\sin C - \sin D \equiv 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \tag{26}$$

Task

Use (7) and (10) to obtain results for the sum and difference of two cosines.

Your solution

Answer

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B \quad \text{and} \quad \cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

$$\therefore \quad \cos(A+B) + \cos(A-B) \equiv 2\cos A\cos B$$

$$\cos(A+B) - \cos(A-B) \equiv -2\sin A\sin B$$

Hence with $C = A+B$ and $D = A-B$

$$\cos C + \cos D \equiv 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \qquad (27)$$

$$\cos C - \cos D \equiv -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \qquad (28)$$

Summary

In this Section we have covered a large number of trigonometric identities. The most important of them and probably the ones most worth memorising are given in the following Key Point.

Key Point 17

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &\equiv 1 \\
\sin 2\theta &\equiv 2\sin\theta\cos\theta \\
\cos 2\theta &\equiv \cos^2\theta - \sin^2\theta \\
&\equiv 2\cos^2\theta - 1 \\
&\equiv 1 - 2\sin^2\theta \\
\sin(A\pm B) &\equiv \sin A\cos B \pm \cos A\sin B \\
\cos(A\pm B) &\equiv \cos A\cos B \mp \sin A\sin B
\end{aligned}$$

Task

A projectile is fired from the ground with an initial speed u m s^{-1} at an angle of elevation $\alpha°$. If air resistance is neglected, the vertical height, y m, is related to the horizontal distance, x m, by the equation

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2} \qquad \text{where } g \text{ m s}^{-2} \text{ is the gravitational constant.}$$

[This equation is derived in HELM 34 Modelling Motion pages 16-17.]

(a) Confirm that $y = 0$ when $x = 0$:

Your solution

Answer
When $y = 0$, the left-hand side of the equation is zero. Since x appears in both of the terms on the right-hand side, when $x = 0$, the right-hand side is zero.

(b) Find an expression for the value of x other than $x = 0$ at which $y = 0$ and state how this value is related to the maximum range of the projectile:

Your solution

Answer

When $y = 0$, the equation can be written $\quad \dfrac{gx^2\sec^2\alpha}{2u^2} - x\tan\alpha = 0$

If $x = 0$ is excluded from consideration, we can divide through by x and rearrange to give

$$\frac{gx\sec^2\alpha}{2u^2} = \tan\alpha$$

To make x the subject of the equation we need to multiply both sides by $\dfrac{2u^2}{g\sec^2\alpha}$.

Given that $1/\sec^2\alpha \equiv \cos^2\alpha$, $\quad \tan\alpha \equiv \sin\alpha/\cos\alpha$ and $\sin 2\alpha \equiv 2\sin\alpha\cos\alpha$, this results in

$$x = \frac{2u^2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

This represents the maximum range.

(c) Find the value of x for which the value of y would be a maximum and thereby obtain an expression for the maximum height:

Your solution

Answer

If air resistance is neglected, we can assume that the parabolic path of the projectile is symmetrical about its highest point. So the highest point will occur at half the maximum range i.e. where

$$x = \frac{u^2 \sin 2\alpha}{2g}$$

Substituting this expression for x in the equation for y gives

$$y = \left(\frac{u^2 \sin 2\alpha}{2g}\right) \tan\alpha - \left(\frac{u^2 \sin 2\alpha}{2g}\right)^2 \frac{g\sec^2\alpha}{2u^2}$$

Using the same trigonometric identities as before,

$$y = \frac{u^2 \sin^2\alpha}{g} - \frac{u^2 \sin^2\alpha}{2g} = \frac{u^2 \sin^2\alpha}{2g}$$ This represents the maximum height.

(d) Assuming $u = 20$ m s^{-1}, $\alpha = 60°$ and $g = 10$ m s^{-2}, find the maximum value of the range and the horizontal distances travelled when the height is 10 m:

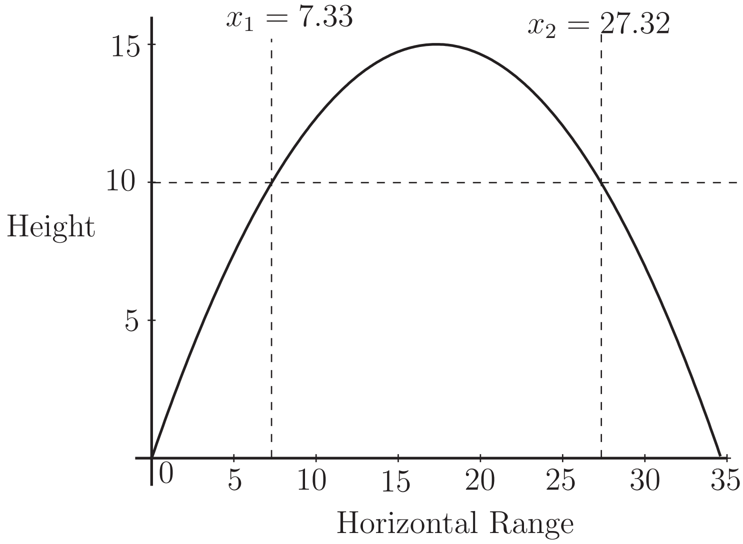
Your solution

Answer

Substitution of $u = 20,\ \alpha = 60,\ g = 10$ and $y = 10$ in the original equation gives a quadratic for x:

$$10 = 1.732x - 0.05x^2 \quad \text{or} \quad 0.05x^2 - 1.732x + 10 = 0$$

Solution of this quadratic yields $x = 7.33$ or $x = 27.32$ as the two horizontal ranges at which $y = 10$. These values are illustrated in the diagram below which shows the complete trajectory of the projectile.



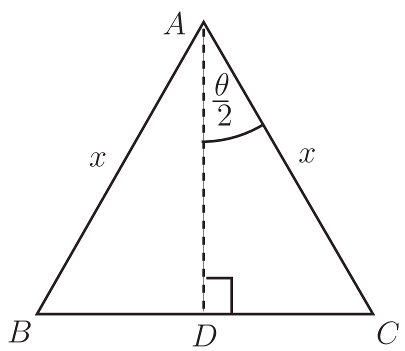
Exercises

1. Show that $\sin t \sec t \equiv \tan t$.
2. Show that $(1+\sin t)(1+\sin(-t)) \equiv \cos^2 t$.
3. Show that $\dfrac{1}{\tan\theta+\cot\theta} \equiv \dfrac{1}{2}\sin 2\theta$.
4. Show that $\sin^2(A+B) - \sin^2(A-B \equiv \sin 2A \sin 2B$.

   (Hint: the left-hand side is the difference of two squared quantities.)
5. Show that $\dfrac{\sin 4\theta+\sin 2\theta}{\cos 4\theta+\cos 2\theta} \equiv \tan 3\theta$.
6. Show that $\cos^4 A - \sin^4 A \equiv \cos 2A$
7. Express each of the following as the sum (or difference) of 2 sines (or cosines)

   (a) $\sin 5x\cos 2x$ (b) $8\cos 6x\cos 4x$ (c) $\dfrac{1}{3}\sin\dfrac{1}{2}x\cos\dfrac{3}{2}x$
8. Express (a) $\sin 3\theta$ in terms of $\cos\theta$. (b) $\cos 3\theta$ in terms of $\cos\theta$.
9. By writing $\cos 4x$ as $\cos 2(2x)$, or otherwise, express $\cos 4x$ in terms of $\cos x$.
10. Show that $\tan 2t \equiv \dfrac{2\tan t}{2-\sec^2 t}$.
11. Show that $\dfrac{\cos 10t - \cos 12t}{\sin 10t + \sin 12t} \equiv \tan t$.
12. Show that the area of an isosceles triangle with equal sides of length x is $\dfrac{x^2}{2}\sin\theta$

    where θ is the angle between the two equal sides. Hint: use the following diagram:



Answers

1. $\sin t.\sec t \equiv \sin t.\dfrac{1}{\cos t} \equiv \dfrac{\sin t}{\cos t} \equiv \tan t.$

2. $(1+\sin t)(1+\sin(-t)) \equiv (1+\sin t)(1-\sin t) \equiv 1-\sin^2 t \equiv \cos^2 t$

3. $\dfrac{1}{\tan\theta+\cos\theta} \equiv \dfrac{1}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}} \equiv \dfrac{1}{\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}} \equiv \dfrac{\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta} \equiv \sin\theta\cos\theta \equiv \dfrac{1}{2}\sin 2\theta$

4. Using the hint and the identity $x^2-y^2 \equiv (x-y)(x+y)$ we have

$$\sin^2(A+B)-\sin^2(A-B) \equiv (\sin(A+B)-\sin(A-B))(\sin(A+B)+\sin(A-B))$$

The first bracket gives

$$\sin A\cos B+\cos A\sin B-(\sin A\cos B-\cos A\sin B) \equiv 2\cos A\sin B$$

Similarly the second bracket gives $2\sin A\cos B.$

Multiplying we obtain $(2\cos A\sin A)(2\cos B\sin B) \equiv \sin 2A.\sin 2B$

5. $\dfrac{\sin 4\theta+\sin 2\theta}{\cos 4\theta+\cos 2\theta} \equiv \dfrac{2\sin 3\theta\cos\theta}{2\cos 3\theta\cos\theta} \equiv \dfrac{\sin 3\theta}{\cos 3\theta} \equiv \tan 3\theta$

6.

$$\begin{aligned}\cos^4 A-\sin^4 A \equiv (\cos A)^4-(\sin A)^4 &\equiv (\cos^2 A)^2-(\sin^2 A)^2\\ &\equiv (\cos^2 A-\sin^2 A)(\cos^2 A+\sin^2 A)\\ &\equiv \cos^2 A-\sin^2 A \equiv \cos 2A\end{aligned}$$

7. (a) Using $\sin A+\sin B \equiv 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

Clearly here $\dfrac{A+B}{2}=5x \qquad \dfrac{A-B}{2}=2x$ giving $A=7x \quad B=3x$

$\therefore \qquad \sin 5x\cos 2x \equiv \dfrac{1}{2}(\sin 7x+\sin 3x)$

(b) Using $\cos A+\cos B \equiv 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right).$

With $\dfrac{A+B}{2}=6x \qquad \dfrac{A-B}{2}=4x$ giving $A=10x \quad B=2x$

$\therefore \quad 8\cos 6x\cos 4x \equiv 4(\cos 6x+\cos 2x)$

(c) $\dfrac{1}{3}\sin\left(\dfrac{1}{2}x\right)\cos\left(\dfrac{3x}{2}\right) \equiv \dfrac{1}{6}(\sin 2x-\sin x)$

Answers

8.

(a)
$$\begin{aligned}\sin 3\theta &\equiv \sin(2\theta+\theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta\\ &\equiv 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta\\ &\equiv 3\sin\theta\cos^2\theta - \sin^3\theta\\ &\equiv 3\sin\theta(1-\sin^2\theta) - \sin^3\theta \equiv 3\sin\theta - 4\sin^3\theta\end{aligned}$$

(b)
$$\begin{aligned}\cos 3\theta &\equiv \cos(2\theta+\theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta\\ &\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta\\ &\equiv \cos^3\theta - 3\sin^2\theta\cos\theta\\ &\equiv \cos^3\theta - 3(1-\cos^2\theta)\cos\theta\\ &\equiv 4\cos^3\theta - 3\cos\theta\end{aligned}$$

9.

$$\begin{aligned}\cos 4x = \cos 2(2x) &\equiv 2\cos^2(2x) - 1\\ &\equiv 2(\cos 2x)^2 - 1\\ &\equiv 2(2\cos^2 x - 1)^2 - 1\\ &\equiv 2(4\cos^4 x - 4\cos^2 x + 1) - 1 \equiv 8\cos^4 x - 8\cos^2 x + 1.\end{aligned}$$

10. $\tan 2t \equiv \dfrac{2\tan t}{1-\tan^2 t} \equiv \dfrac{2\tan t}{1-(\sec^2 t - 1)} \equiv \dfrac{2\tan t}{2-\sec^2 t}$

11. $\cos 10t - \cos 12t \equiv 2\sin 11t\sin t \qquad \sin 10t + \sin 12t \equiv 2\sin 11t\cos(-t)$

$$\therefore \quad \frac{\cos 10t - \cos 12t}{\sin 10t + \sin 12t} \equiv \frac{\sin t}{\cos(-t)} \equiv \frac{\sin t}{\cos t} \equiv \tan t$$

12. The right-angled triangle ACD has area $\dfrac{1}{2}(CD)(AD)$

But $\quad \sin\left(\dfrac{\theta}{2}\right) = \dfrac{CD}{x} \qquad \therefore \quad CD = x\sin\left(\dfrac{\theta}{2}\right)$

$\quad \cos\left(\dfrac{\theta}{2}\right) = \dfrac{AD}{x} \qquad \therefore \quad AD = x\cos\left(\dfrac{\theta}{2}\right)$

$\therefore$ area of $\Delta ACD = \dfrac{1}{2}x^2\sin\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{\theta}{2}\right) = \dfrac{1}{4}x^2\sin\theta$

$\therefore$ area of $\Delta ABC = 2 \times$ area of $\Delta ACD = \dfrac{1}{2}x^2\sin\theta$