Trigonometric Functions

4.2

Introduction

Our discussion so far has been limited to right-angled triangles where, apart from the right-angle itself, all angles are necessarily less than $90°$. We now extend the definitions of the trigonometric functions to any size of angle, which greatly broadens the range of applications of trigonometry.

Prerequisites

Before starting this Section you should ...

- have a basic knowledge of the geometry of triangles

Learning Outcomes

On completion you should be able to ...

- express angles in radians
- define trigonometric functions generally
- sketch the graphs of the three main trigonometric functions: $\sin$, $\cos$, $\tan$

1. Trigonometric functions for any size angle

The radian

First we introduce an alternative to measuring angles in degrees. Look at the circle shown in Figure 19(a). It has radius r and we have shown an arc AB of length ℓ (measured in the same units as r.) As you can see the arc subtends an angle θ at the centre O of the circle.

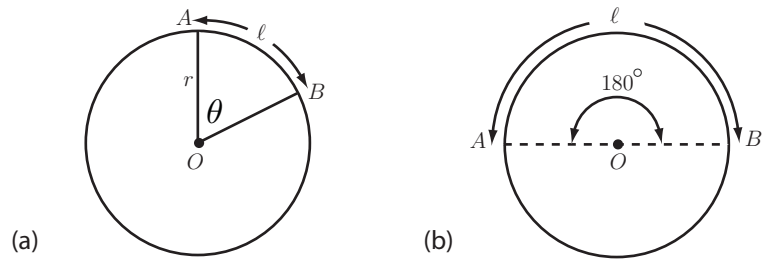


Figure 19

The angle θ in **radians** is defined as

$$\theta = \frac{\text{length of arc } AB}{\text{radius}} = \frac{\ell}{r}$$

So, for example, if $r = 10$ cm, $\ell = 20$ cm, the angle θ would be $\dfrac{20}{10} = 2$ radians.

The relation between the value of an angle in radians and its value in degrees is readily obtained as follows. Referring to Figure 19(b) imagine that the arc AB extends to cover half the complete perimeter of the circle. The arc length is now πr (half the circumference of the circle) so the angle θ subtended by AB is now

$$\theta = \frac{\pi r}{r} = \pi \quad \text{radians}$$

But clearly this angle is $180°$. Thus π radians is the same as $180°$.

Note conversely that since π radians $= 180°$ then 1 radian $= \dfrac{180}{\pi}$ degrees (about $57.3°$).

Key Point 6

$180° = \pi$ radians

$360° = 2\pi$ radians

$1° = \dfrac{\pi}{180}$ radians

$x° = \dfrac{\pi x}{180}$ radians

1 radian $= \dfrac{180}{\pi}$ degrees ($\approx 57.3°$)

y radians $= \dfrac{180y}{\pi}$ degrees

Write down the values in radians of $30°,\ 45°,\ 90°,\ 135°$. (Leave your answers as multiples of π.)

Your solution

Answer

$30° = \pi \times \dfrac{30}{180} = \dfrac{\pi}{6}$ radians $\quad 45° = \dfrac{\pi}{4}$ radians $\quad 90° = \dfrac{\pi}{2}$ radians $\quad 135° = \dfrac{3\pi}{4}$ radians

Write in degrees the following angles given in radians

$$\frac{\pi}{10},\quad \frac{\pi}{5},\quad \frac{7\pi}{10},\quad \frac{23\pi}{12}$$

Your solution

Answer

$\dfrac{\pi}{10}$ rad $= \dfrac{180}{\pi} \times \dfrac{\pi}{10} = 18°$ $\quad \dfrac{\pi}{5}$ rad $= \dfrac{180}{\pi} \times \dfrac{\pi}{5} = 36°$ $\quad \dfrac{7\pi}{10}$ rad $= \dfrac{180}{\pi} \times \dfrac{7\pi}{10} = 126°$

$\dfrac{23\pi}{12}$ rad $= \dfrac{180}{\pi} \times \dfrac{23\pi}{12} = 345°$

Put your calculator into **radian mode** (using the DRG button if necessary) for this Task: Verify these facts by first converting the angles to radians:

$\sin 30° = \dfrac{1}{2}$ $\quad \cos 45° = \dfrac{1}{\sqrt{2}}$ $\quad \tan 60° = \sqrt{3}$ (Use the π button to obtain π.)

Your solution

Answer

$$\sin 30° = \sin\left(\frac{\pi}{6}\right) = 0.5, \qquad \cos 45° = \cos\left(\frac{\pi}{4}\right) = 0.7071 = \frac{1}{\sqrt{2}},$$

$$\tan 60° = \tan\left(\frac{\pi}{3}\right) = 1.7320 = \sqrt{3}$$

2. General definitions of trigonometric functions

We now define the trigonometric functions in a more general way than in terms of ratios of sides of a right-angled triangle. To do this we consider a circle of **unit radius** whose centre is at the origin of a Cartesian coordinate system and an arrow (or **radius vector**) OP from the centre to a point P on the circumference of this circle. We are interested in the angle θ that the arrow makes with the **positive** x-axis. See Figure 20.

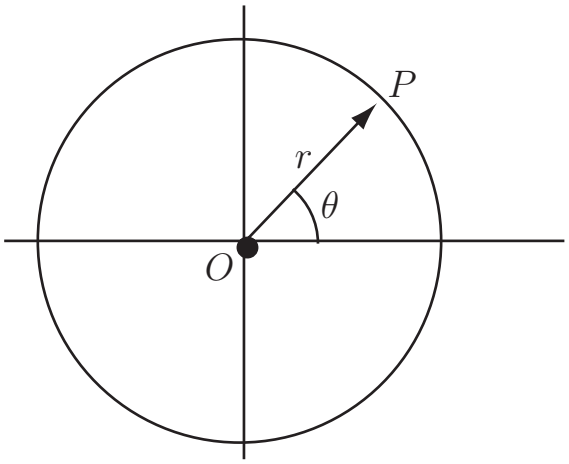
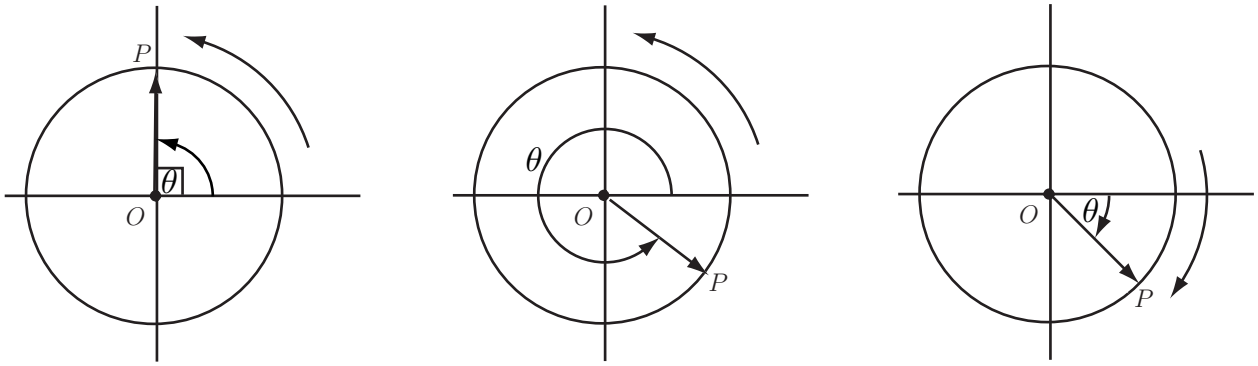


Figure 20

Imagine that the vector OP rotates in **anti-clockwise direction**. With this sense of rotation the angle θ is taken as positive whereas a **clockwise** rotation is taken as negative. See examples in Figure 21.



$\theta = 90^\circ = \dfrac{\pi}{2}$ rad $\qquad$ $\theta = 315^\circ = \dfrac{7\pi}{4}$ rad $\qquad$ $\theta = -45^\circ = -\dfrac{\pi}{4}$ rad

Figure 21

The sine and cosine of an angle

For $0 \leq \theta \leq \dfrac{\pi}{2}$ (called the **first** quadrant) we have the following situation with our unit radius circle. See Figure 22.

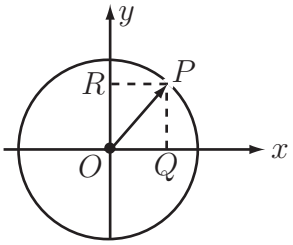


Figure 22

The **projection** of OP along the positive $x-$axis is OQ. But, in the right-angled triangle OPQ

$$\cos\theta = \frac{OQ}{OP} \qquad \text{or} \qquad OQ = OP\cos\theta$$

and since OP has unit length $\qquad \cos\theta = OQ \qquad (3)$

Similarly in this right-angled triangle

$$\sin\theta = \frac{PQ}{OP} \qquad \text{or} \qquad PQ = OP\sin\theta$$

but $PQ = OR$ and OP has unit length
so $$\sin\theta = OR \qquad (4)$$

Equation (3) tells us that we can interpret $\mathbf{cos}\,\theta$ as the projection of OP along the **positive x-axis** and $\mathbf{sin}\,\theta$ as the projection of OP along the **positive y-axis**.
We shall use these interpretations as the **definitions** of $\sin\theta$ and $\cos\theta$ for **any** values of θ.

Key Point 7

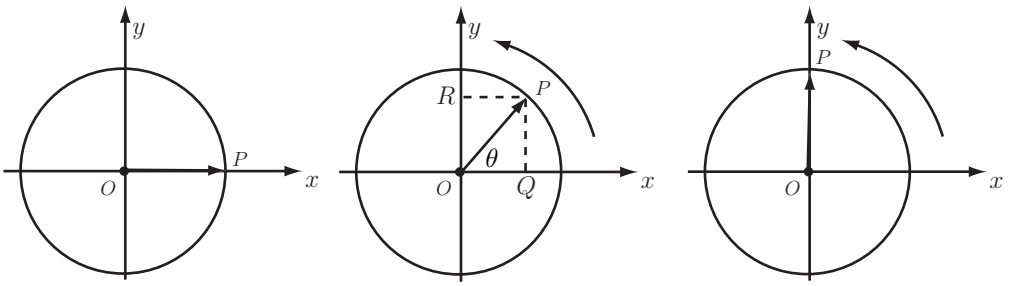
For a radius vector OP of a circle of unit radius making an angle θ with the positive $x-$axis

$\cos\theta =$ projection of OP along the positive $x-$axis

$\sin\theta =$ projection of OP along the positive $y-$axis

Sine and cosine in the four quadrants

First quadrant $(0 \leq \theta \leq 90°)$



$\theta = 0°$
$OQ = OP = 1$
$\therefore \cos 0° = 1$
$OR = 0$
$\therefore \sin 0° = 0$

$0 < \theta < 90°$
$\cos\theta = OQ$
$\therefore \ 0 < \cos\theta < 1$
$\sin\theta = OR$
$\therefore \ 0 < \sin\theta < 1$

$\theta = 90°$
$OQ = 0$
$\therefore \cos 90° = 0$
$OR = OP = 1$
$\therefore \sin 90° = 1$

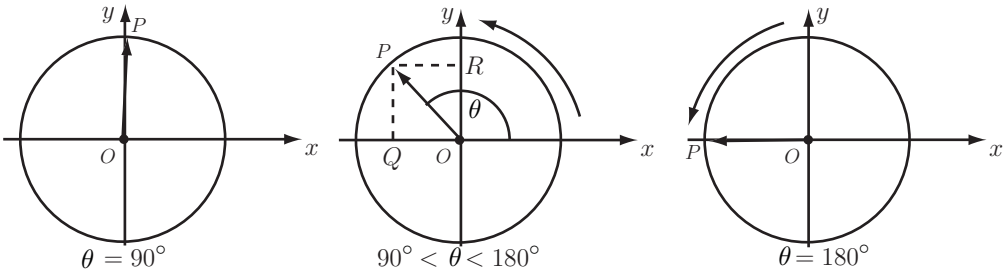
Figure 23

It follows from Figure 23 that $\cos\theta$ decreases from 1 to 0 as OP rotates from the horizontal position to the vertical, i.e. as θ increases from $0°$ to $90°$.

$\sin\theta = OR$ increases from 0 (when $\theta = 0$) to 1 (when $\theta = 90°$).

Second quadrant $(90° \leq \theta \leq 180°)$

Referring to Figure 24, remember that it is the projections along the **positive** x and y axes that are used to define $\cos\theta$ and $\sin\theta$ respectively. It follows that as θ increases from $90°$ to $180°$, $\cos\theta$ decreases from 0 to -1 and $\sin\theta$ decreases from 1 to 0.



$\theta = 90°$
$\cos 90° = 0$
$\sin 90° = 1$

$90° < \theta < 180°$
$\cos\theta = OQ$ (negative)
$\sin\theta = OR$ (positive)

$\theta = 180°$
$\cos\theta = OQ = OP = -1$
$\sin\theta = OR = 0$

Figure 24

Considering for example an angle of 135°, referring to Figure 25, by symmetry we have:

$$\sin 135° = OR = \sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 135° = OQ_2 = -OQ_1 = -\cos 45° = -\frac{1}{\sqrt{2}}$$

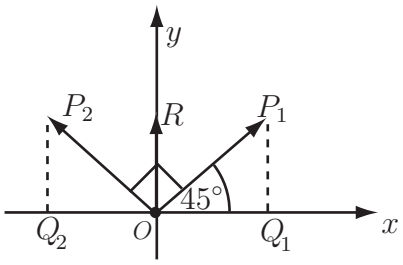


Figure 25

Key Point 8

$\sin(180 - x) \equiv \sin x$ and $\cos(180 - x) \equiv -\cos x$

Without using a calculator write down the values of

$$\sin 120^\circ,\ \sin 150^\circ,\ \cos 120^\circ,\ \cos 150^\circ,\ \tan 120^\circ,\ \tan 150^\circ.$$

(Note that $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ for any value of θ.)

Your solution

Answer

$$\sin 120^\circ = \sin(180 - 60) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 150^\circ = \sin(180 - 30) = \sin 30^\circ = \frac{1}{2}$$

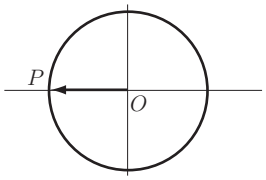
$$\cos 120^\circ = -\cos 60 = -\frac{1}{2}$$

$$\cos 150^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\tan 120^\circ = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

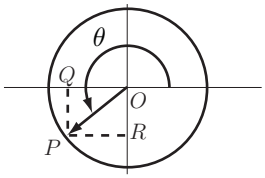
$$\tan 150^\circ = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

Third quadrant $(180° \leq \theta \leq 270°)$.



$\cos 180° = -1$

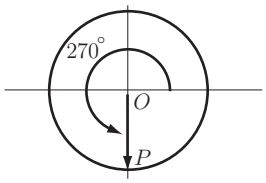
$\sin 180° = 0$



$180° < \theta < 270°$

$\cos\theta = OQ$ (negative)

$\sin\theta = OR$ (negative)



$\theta = 270°$

$\cos\theta = ?$

$\sin\theta = ?$

Figure 26

Using the projection definition write down the values of $\cos 270°$ and $\sin 270°$.

Your solution

Answer

$\cos 270° = 0$ (OP has zero projection along the positive $x-$axis)

$\sin 270° = -1$ (OP is directed along the negative axis)

Thus in the third quadrant, as θ increases from $180°$ to $270°$ so $\cos\theta$ increases from -1 to 0 whereas $\sin\theta$ decreases from 0 to -1.

From the results of the last Task, with $\theta = 180° + x$ (see Figure 27) we obtain for all x the relations:

$$\sin\theta = \sin(180 + x) = OR = -OR' = -\sin x \qquad \cos\theta = \cos(180 + x) = OQ = -OQ' = -\cos x$$

Hence $\tan(180 + x) = \dfrac{\sin(180° + x)}{\cos(180° + x)} = \dfrac{\sin x}{\cos x} = +\tan x$ for all x.

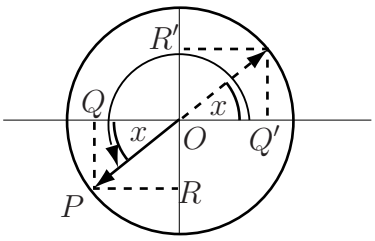
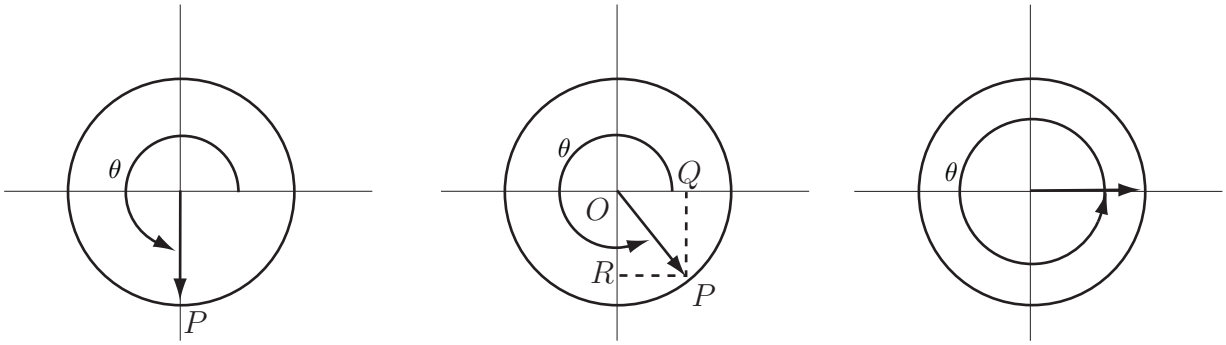


Figure 27: $\theta = 180° + x$

Key Point 9

$\sin(180 + x) \equiv -\sin x$ $\qquad$ $\cos(180 + x) \equiv -\cos x$ $\qquad$ $\tan(180 + x) \equiv +\tan x$

Fourth quadrant $(270° \leq \theta \leq 360°)$



$\theta = 270°$
$\cos\theta = 0$
$\sin\theta = -1$

$270° < \theta < 360°$
(alternatively $-90° < \theta < 0°$)
$\cos\theta = OQ > 0$
$\sin\theta = OR < 0$

$360°$
(results as for $0°$)

Figure 28

From Figure 28 the results in Key Point 10 should be clear.

Key Point 10

$$\cos(-x) \equiv \cos x \qquad \sin(-x) \equiv -\sin x \qquad \tan(-x) \equiv -\tan x.$$

Write down (without using a calculator) the values of
$\sin 300°, \quad \sin(-60°), \quad \cos 330°, \quad \cos(-30°).$

Describe the behaviour of $\cos\theta$ and $\sin\theta$ as θ increases from $270°$ to $360°$.

Your solution

Answer

$$\sin 300° = -\sin 60° = -\sqrt{3}/2 \qquad \cos 330° = \cos 30° = \sqrt{3}/2$$

$$\sin(-60°) = -\sin 60° = -\sqrt{3}/2 \qquad \cos(-30°) = \cos 30° = \sqrt{3}/2$$

$\cos\theta$ increases from 0 to 1 and $\sin\theta$ increases from -1 to 0 as θ increases from $270°$ to $360°$.

Rotation beyond the fourth quadrant $(360^\circ < \theta)$

If the vector OP continues to rotate around the circle of unit radius then in the next complete rotation θ increases from 360° to 720°. However, a θ value of, say, 405° is indistinguishable from one of 45° (just one extra complete revolution is involved).

So $\sin(405^\circ) = \sin 45^\circ = \dfrac{1}{\sqrt{2}}$ and $\cos(405^\circ) = \cos 45^\circ = \dfrac{1}{\sqrt{2}}$

In general $\sin(360^\circ + x^\circ) = \sin x^\circ, \qquad \cos(360^\circ + x^\circ) = \cos x^\circ$

Key Point 11

If n is any integer $\sin(x^\circ + 360n^\circ) \equiv \sin x^\circ \qquad \cos(x^\circ + 360n^\circ) \equiv \cos x^\circ$

or, since $360^\circ \equiv 2\pi$ radians, $\sin(x + 2n\pi) \equiv \sin x \qquad \cos(x + 2n\pi) = \cos x$

We say that the functions $\sin x$ and $\cos x$ are **periodic** with period (in radian measure) of 2π.

3. Graphs of trigonometric functions

Graphs of $\sin\theta$ and $\cos\theta$

Since we have defined both $\sin\theta$ and $\cos\theta$ in terms of the projections of the radius vector OP of a circle of **unit radius** it follows immediately that

$$-1 \leq \sin\theta \leq +1 \qquad \text{and} \qquad -1 \leq \cos\theta \leq +1 \qquad \text{for any value of } \theta.$$

We have discussed the behaviour of $\sin\theta$ and $\cos\theta$ in each of the four quadrants in the previous subsection.

Using all the above results we can draw the graphs of these two trigonometric functions. See Figure 29. We have labelled the horizontal axis using radians and have shown two **periods** in each case.

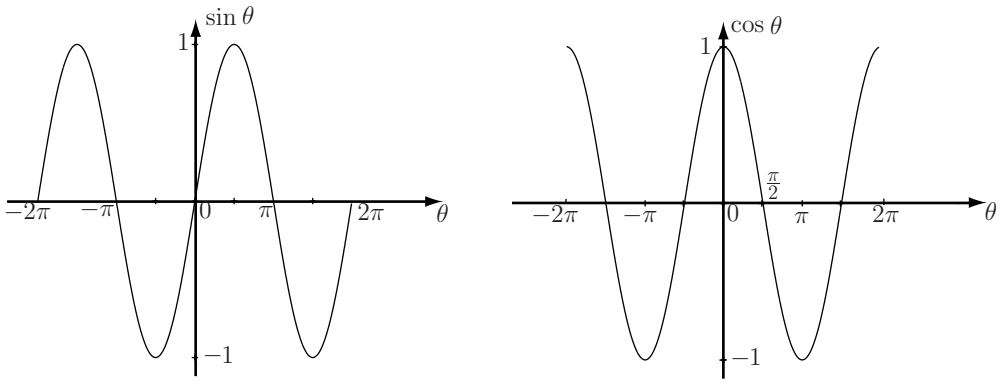


Figure 29

We have extended the graphs to negative values of θ using the relations $\sin(-\theta) = \sin\theta$, $\cos(-\theta) = \cos\theta$. Both graphs could be extended indefinitely to the left ($\theta \to -\infty$) and right ($\theta \to +\infty$).

(a) Using the graphs in Figure 29 and the fact that $\tan\theta \equiv \sin\theta/\cos\theta$ calculate the values of $\tan 0$, $\tan\pi$, $\tan 2\pi$.
(b) For what values of θ is $\tan\theta$ undefined?
(c) State whether $\tan\theta$ is positive or negative in each of the four quadrants.

Your solution

(a)

(b)

(c)

Answer

(a)

$$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

$$\tan\pi = \frac{\sin\pi}{\cos\pi} = \frac{0}{-1} = 0$$

$$\tan 2\pi = \frac{\sin 2\pi}{\cos 2\pi} = \frac{0}{1} = 0$$

(b)

$\tan\theta$ is not be defined when $\cos\theta = 0$ i.e. when $\theta = \pm\ \dfrac{\pi}{2},\ \pm\ \dfrac{3\pi}{2}, \pm\ \dfrac{5\pi}{2}, \ldots$

(c)

1st quadrant: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{+\text{ve}}{+\text{ve}} = +\text{ve}$

2nd quadrant: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{+\text{ve}}{-\text{ve}} = -\text{ve}$

3rd quadrant: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\text{ve}}{-\text{ve}} = +\text{ve}$

4th quadrant: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\text{ve}}{+\text{ve}} = -\text{ve}$

The graph of $\tan\theta$

The graph of $\tan\theta$ against θ, for $-2\pi \leq \theta \leq 2\pi$ is then as in Figure 30. Note that whereas $\sin\theta$ and $\cos\theta$ have period 2π, $\tan\theta$ has period π.

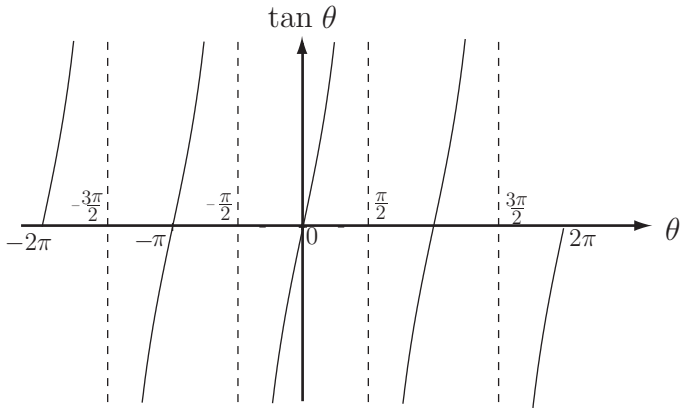
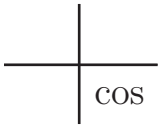


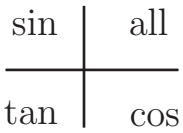
Figure 30

On the following diagram showing the four quadrants mark which trigonometric quantities $\cos,\ \sin,\ \tan,$ are positive in the four quadrants. One entry has been made already.

Your solution



Answer



Engineering Example 4

Optical interference fringes due to a glass plate

Monochromatic light of intensity I_0 propagates in air before impinging on a glass plate (see Figure 31). If a screen is placed beyond the plate then a pattern is observed including alternate light and dark regions. These are **interference** fringes.

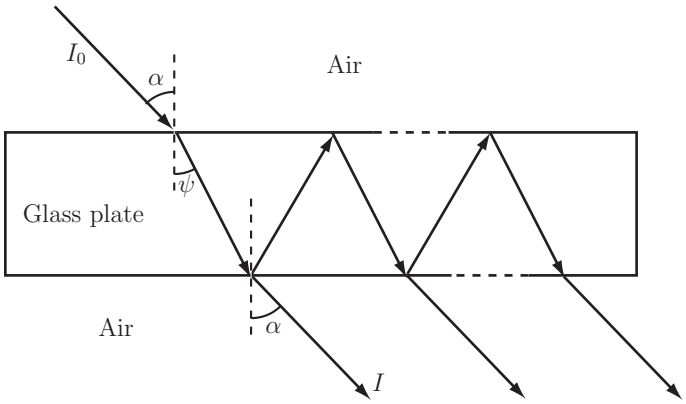


Figure 31: Geometry of a light ray transmitted and reflected through a glass plate

The intensity I of the light wave transmitted through the plate is given by

$$I = \frac{I_0|t|^4}{1 + |r|^4 - 2|r|^2 \cos\theta}$$

where t and r are the complex transmission and reflection coefficients. The phase angle θ is the sum of

(i) a phase proportional to the incidence angle α and

(ii) a fixed phase lag due to multiple reflections.

The problem is to establish the form of the intensity pattern (i.e. the minima and maxima characteristics of interference fringes due to the plate), and deduce the shape and position θ of the fringes captured by a screen beyond the plate.

Solution

The intensity of the optical wave outgoing from the glass plate is given by

$$I = \frac{I_0|t|^4}{1 + |r|^4 - 2|r|^2 \cos\theta} \tag{1}$$

The light intensity depends solely on the variable θ as shown in equation (1), and the objective is to find the values θ that will minimize and maximize I. The angle θ is introduced in equation (1) through the function $\cos\theta$ in the denominator. We consider first the maxima of I.

Solution (contd.)

Light intensity maxima

I is maximum when the denominator is minimum. This condition is obtained when the factor $2|r|\cos\theta$ is maximum due to the minus sign in the denominator. As stated in Section 4.2, the maxima of $2|r|\cos\theta$ occur when $\cos\theta = +1$. Values of $\cos\theta = +1$ correspond to $\theta = 2n\pi$ where $n = \ldots -2, -1, 0, 1, 2, \ldots$ (see Section 4.5) and θ is measured in radians. Setting $\cos\theta = +1$ in equation (1) gives the intensity maxima

$$I_{\max} = \frac{I_0|t|^4}{1 + |r|^4 - 2|r|^2}.$$

Since the denominator can be identified as the square of $(1 + |r|^2)$, the final result for maximum intensity can be written as

$$I_{\max} = \frac{I_0|t|^4}{(1 - |r|^2)^2}. \tag{2}$$

Light intensity minima

I is minimum when the denominator in (1) is maximum. As a result of the minus sign in the denominator, this condition is obtained when the factor $2|r|\cos\theta$ is minimum. The minima of $2|r|\cos\theta$ occur when $\cos\theta = -1$. Values of $\cos\theta = -1$ correspond to $\theta = \pi(2n + 1)$ where $n = \ldots -2, -1, 0, 1, 2, \ldots$ (see Section 4.5). Setting $\cos\theta = -1$ in equation (1) gives an expression for the intensity minima

$$I_{\min} = \frac{I_0|t|^4}{1 + |r|^4 + 2|r|^2}.$$

Since the denominator can be recognised as the square of $(1 + |r|^2)$, the final result for minimum intensity can be written as

$$I_{\min} = \frac{I_0|t|^4}{(1 + |r|^2)^2} \tag{3}$$

Interpretation

The interference fringes for intensity maxima or minima occur at constant angle θ and therefore describe concentric rings of alternating light and shadow as sketched in the figure below. From the centre to the periphery of the concentric ring system, the fringes occur in the following order

(a) a fringe of maximum light at the centre (bright dot for $\theta = 0$),

(b) a circular fringe of minimum light at angle $\theta = \pi$,

(c) a circular fringe of maximum light at 2π etc.

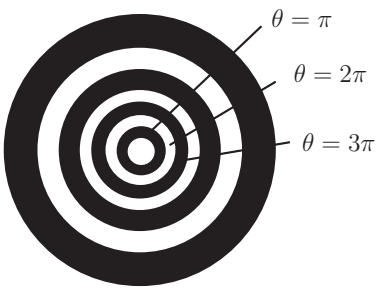


Figure 32: Sketch of interference fringes due to a glass plate

Exercises

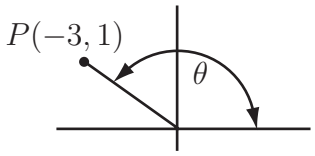
1. Express the following angles in radians (as multiples of π)

   (a) 120° (b) 20° (c) 135° (d) 300° (e) -90° (f) 720°

2. Express in degrees the following quantities which are in radians

   (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{6}$ (d) $\frac{11\pi}{9}$ (e) $-\frac{\pi}{8}$ (f) $\frac{1}{\pi}$

3. Obtain the **precise** values of all 6 trigonometric functions of the angle θ for the situation shown in the figure:



4. Obtain all the values of x between 0 and 2π such that

   (a) $\sin x = \frac{1}{\sqrt{2}}$ (b) $\cos x = \frac{1}{2}$ (c) $\sin x = -\frac{\sqrt{3}}{2}$ (d) $\cos x = -\frac{1}{\sqrt{2}}$ (e) $\tan x = 2$

   (f) $\tan x = -\frac{1}{2}$ (g) $\cos(2x + 60^\circ) = 2$ (h) $\cos(2x + 60^\circ) = \frac{1}{2}$

5. Obtain all the values of θ in the given domain satisfying the following quadratic equations

   (a) $2\sin^2\theta - \sin\theta = 0 \qquad 0 \le \theta \le 360^\circ$

   (b) $2\cos^2\theta + 7\cos\theta + 3 = 0 \qquad 0 \le \theta \le 360^\circ$

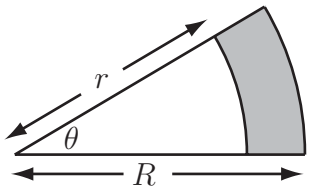
   (c) $4\sin^2\theta - 1 = 0$

6. (a) Show that the area A of a sector formed by a central angle θ radians in a circle of radius r is given by

   $$A = \frac{1}{2}r^2\theta.$$

   (Hint: By proportionality the ratio of the area of the sector to the total area of the circle equals the ratio of θ to the total angle at the centre of the circle.)

   (b) What is the value of the shaded area shown in the figure if θ is measured (i) in radians, (ii) in degrees?



7. Sketch, over $0 < \theta < 2\pi$, the graph of (a) $\sin 2\theta$ (b) $\sin\frac{1}{2}\theta$ (c) $\cos 2\theta$ (d) $\cos\frac{1}{2}\theta$.

   Mark the horizontal axis in radians in each case. Write down the period of $\sin 2\theta$ and the period of $\cos\frac{1}{2}\theta$.

Answers

1. (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{9}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{3}$ (e) $-\frac{\pi}{2}$ (f) 4π

2. (a) 15° (b) 270° (c) 150° (d) 220° (e) -22.5° (f) $\frac{180^\circ}{\pi^2}$

3. The distance of the point P from the origin is $r = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$. Then, since P lies on a circle radius $\sqrt{10}$ rather than a circle of unit radius:

$$\sin\theta = \frac{1}{\sqrt{10}} \qquad \text{cosec}\,\theta = \sqrt{10}$$

$$\cos\theta = -\frac{3}{\sqrt{10}} \qquad \sec\theta = -\frac{\sqrt{10}}{3}$$

$$\tan\theta = \frac{1}{-3} = -\frac{1}{3} \qquad \cot\theta = -3$$

4. (a) $x = 45^\circ \left(\frac{\pi}{4} \text{ radians}\right)$ $x = 135^\circ \left(\frac{3\pi}{4}\right)$ (recall $\sin(180 - x) = \sin x$)

   (b) $x = 60^\circ \left(\frac{\pi}{3}\right)$ $x = 300^\circ \left(\frac{5\pi}{3}\right)$

   (c) $x = 240^\circ \left(\frac{4\pi}{3}\right)$ $x = 300^\circ \left(\frac{5\pi}{3}\right)$

   (d) $x = 135^\circ \left(\frac{3\pi}{4}\right)$ $x = 225^\circ \left(\frac{5\pi}{4}\right)$

   (e) $x = 63.43^\circ$ $x = 243.43^\circ$ (remember $\tan x$ has period 180° or π radians)

   (f) $x = 153.43^\circ$ $x = 333.43^\circ$

   (g) No solution !

   (h) $x = 0^\circ,\ 120^\circ,\ 180^\circ,\ 300^\circ,\ 360^\circ$

5. (a) $2\sin^2\theta - \sin\theta = 0$ so $\sin\theta(2\sin\theta - 1) = 0$ so $\sin\theta = 0$
   giving $\theta = 0^\circ,\ 180^\circ,\ 360^\circ$ or $\sin\theta = \frac{1}{2}$ giving $\theta = 30^\circ,\ 150^\circ$

   (b) $2\cos^2\theta + 7\cos\theta + 3 = 0$. With $x = \cos\theta$ we have $2x^2 + 7x + 3 = 0$ $(2x+1)(x+3) = 0$ (factorising) so $2x = -1$ or $x = -\frac{1}{2}$. The solution $x = -3$ is impossible since $x = \cos\theta$.

   The equation $x = \cos\theta = -\frac{1}{2}$ has solutions $\theta = 120^\circ,\ 240^\circ$

   (c) $4\sin^2\theta = 1$ so $\sin^2\theta = \frac{1}{4}$ i.e. $\sin\theta = \pm\frac{1}{2}$ giving $\theta = 30^\circ,\ 150^\circ,\ 210^\circ,\ 330^\circ$

Answers continued

6. (a) Using the hint,

$$\frac{\theta}{2\pi} = \frac{A}{\pi r^2}$$

from where we obtain $A = \dfrac{\pi r^2 \theta}{2\pi} = \dfrac{r^2\theta}{2}$

(b) With θ in radians the shaded area is

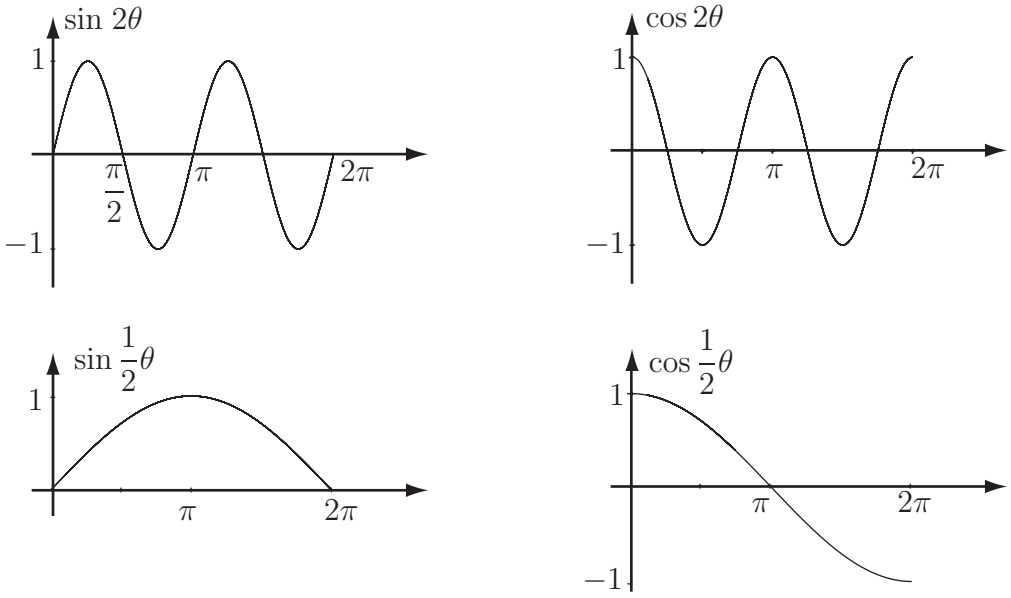
$$S = \frac{R^2\theta}{2} - \frac{r^2\theta}{2} = \frac{\theta}{2}(R^2 - r^2)$$

If θ is in degrees, then since x radians $= \dfrac{180x^\circ}{\pi}$ or $x^\circ = \dfrac{\pi x}{180}$ radians, we have

$$S = \frac{\pi\theta^\circ}{360^\circ}(R^2 - r^2)$$

7. The graphs of $\sin 2\theta$ and $\cos 2\theta$ are identical in form with those of $\sin\theta$ and $\cos\theta$ respectively but oscillate twice as rapidly.

The graphs of $\sin\dfrac{1}{2}\theta$ and $\cos\dfrac{1}{2}\theta$ oscillate half as rapidly as those of $\sin\theta$ and $\cos\theta$.



From the graphs $\sin 2\theta$ has period 2π and $\cos\frac{1}{2}\theta$ has period 4π. In general $\sin n\theta$ has period $2\pi/n$.