MATHEMATICS

SUPPORT CENTRE

Title: Partial Fractions

Target: On completion of this worksheet you should be able to express any proper algebraic fraction as a sum of its partial fractions.

When we add algebraic fractions we find a common denominator, express each of the fractions with this common denominator and then add the numerators.

See the algebra sheet on algebraic fractions and operations if you need to revise this.

Sometimes it is useful to do the reverse of this process. That is when we write an algebraic fraction as a sum of its **partial fractions.**

In order to do this we first need to consider the possible denominators of the partial fractions. We should:

- Factorise completely the denominator of the original fraction.
- List the terms of the factorisation.
- If there are any repeated terms include all possible powers of the term.

The terms in the list are our possible denominators.

Example.
Find the denominators of the partial fractions of $\dfrac{3x+2}{x^2+5x+4}$ and $\dfrac{2}{x^3-4x^2+4x}$.

1. $x^2+5x+4=(x+4)(x+1)$.
   Possible denominators are $(x + 4)$ and $(x + 1)$.
2. $x^3-4x^2+4x = x(x^2-4x+4)$
   $= x(x-2)(x-2)$.
   Possible denominators are x, $(x - 2)$ and $(x - 2)^2$.

Sometimes we cannot factorise the denominator into linear factors. This does not effect the method of finding the possible denominators of the partial fractions.

Example.
Find the denominators of the partial fractions of $\dfrac{5}{x^3+2x}$.

$x^3+2x = x(x^2+2)$. We cannot factorise this further.
The denominators of the partial fractions are x and $(x^2 + 2)$.

Exercise.
Find the denominators of the partial fractions of the following:

1. $\dfrac{3}{x(x+4)}$.
2. $\dfrac{x}{9x^2-16}$.
3. $\dfrac{2x-3}{x^2-3x-54}$.
4. $\dfrac{2}{s(s^2+1)}$.
5. $\dfrac{3x}{x^2+6x+9}$.
6. $\dfrac{-7}{x^3+9x}$.

(Answers:{ x, $(x+4)$}; {$(3x-4)$, $(3x+4)$}; {$(x-9)$, $(x+6)$}; {s, (s^2+1)}; {$(x+3)$,$(x+3)^2$} {x, (x^2+9)}.)

Once we have found the denominators of the partial fractions it remains to find the numerators. We must consider what the possibilities are.

An algebraic fraction is said to be **proper** if the highest power of the variable in the numerator is less than the highest power of the variable in the denominator.

Examples

$$\frac{2x}{x^2+4x+2},\ \frac{3x+1}{x^3}$$

are proper algebraic fractions.

$$\frac{x^2}{3x^2+4x+2},\ \frac{3x+1}{7x+6}$$

are not proper algebraic fractions.

To find the possibilities of the numerators of the partial fractions we will use the property that if an algebraic fraction is proper then its partial fractions must be proper.

Therefore if the denominator of the partial fraction is:

- Linear, the numerator must be a constant.
- Quadratic, the numerator must be linear.

We denote constants by capital letters, say *A*, *B*,*C*, etc.

Linear expressions are of the form *Bx* + *C* where *B* and *C* are just constants.

We must not use a letter more than once.

Example.

Express as a sum of partial fractions

$$\frac{3}{(x+2)(x+3)} \text{ and } \frac{5}{s(s^2+1)}.$$

1. $\frac{3}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}.$
2. $\frac{5}{s(s^2+1)} = \frac{A}{s} + \frac{Bx+C}{s^2+1}.$

It remains to find what the unknown constants are.

To find the unknown constants we must add the partial fractions we can then equate the numerators of the original fraction and the sum of the partial fractions.

Example.

We will complete the previous example.

$$\frac{3}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$
$$= \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}.$$

Therefore:

$$3 = A(x+3)+B(x+2).$$

Letting *x* = -2 gives 3 = *A*.

Letting *x* = -3 gives 3 = -*B*, hence *B* = -3.

So substituting these values into the original partial fractions gives

$$\frac{3}{(x+2)(x+3)} = \frac{3}{x+2} + \frac{-3}{x+3}.$$

If you have difficulty with this refer to the algebra sheet on polynomials.

Exercise.

Express the following as a sum of partial fractions:

1. $\frac{6}{(x-6)(x-2)}.$
2. $\frac{4x}{(x+1)(x-3)}.$
3. $\frac{3x+1}{(x+2)(x-5)}.$
4. $\frac{x+2}{(x+1)(x^2-9)}.$
5. $\frac{1}{(x-2)(x^2+1)}.$
6. $\frac{x-2}{(x^2+2)(2x+1)}.$
7. $\frac{2x+5}{(x+2)^2(2x+1)}.$

(Answers: $\frac{3}{2(x-6)} - \frac{3}{2(x-2)}$, $\frac{1}{x+1} + \frac{3}{x-3}$, $\frac{5}{7(x+2)} + \frac{16}{7(x-5)}$, $\frac{-1}{8(x+1)} + \frac{5}{24(x-3)} - \frac{1}{12(x+3)}$, $\frac{1}{5(x-2)} + \frac{(-x-2)}{5(x^2+1)}$, $\frac{5x+2}{9(x^2+2)} - \frac{10}{9(2x+1)}$, $\frac{-8}{9(x+2)} - \frac{1}{3(x+2)^2} + \frac{16}{9(2x+1)}$.)