Solving Quadratic Equations

3.2

Introduction

A **quadratic equation** is one which can be written in the form $ax^2 + bx + c = 0$ where a, b and c are numbers, $a \neq 0$, and x is the unknown whose value(s) we wish to find. In this Section we describe several ways in which quadratic equations can be solved.

Prerequisites

Before starting this Section you should ...

- be able to solve linear equations

Learning Outcomes

On completion you should be able to ...

- recognise a quadratic equation
- solve a quadratic equation by factorisation
- solve a quadratic equation using the standard formula
- solve a quadratic equation by completing the square
- interpret the solution of a quadratic equation graphically

1. Quadratic equations

Key Point 3

A **quadratic equation** is one which can be written in the form

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

where a, b and c are given numbers and x is the unknown whose value(s) must be found.

For example

$$2x^2 + 7x - 3 = 0, \qquad x^2 + x + 1 = 0, \qquad 0.5x^2 + 3x + 9 = 0$$

are all quadratic equations. To ensure the presence of the x^2 term, the number a, in the general expression $ax^2 + bx + c$ cannot be zero. However b or c may be zero, so that

$$4x^2 + 3x = 0, \quad 2x^2 - 3 = 0 \quad \text{and} \quad 6x^2 = 0$$

are also quadratic equations. Frequently, quadratic equations occur in non-standard form but where necessary they can be rearranged into standard form. For example

$$3x^2 + 5x = 8, \qquad \text{can be re-written as} \qquad 3x^2 + 5x - 8 = 0$$

$$2x^2 = 8x - 9, \qquad \text{can be re-written as} \qquad 2x^2 - 8x + 9 = 0$$

$$1 + x = \frac{1}{x}, \qquad \text{can be re-written as} \qquad x^2 + x - 1 = 0$$

To solve a quadratic equation we must find values of the unknown x which make the left-hand and right-hand sides equal. Such values are known as **solutions** or **roots** of the quadratic equation.

Note the difference between solving quadratic equations in comparison to solving linear equations. A quadratic equation will generally have **two** values of x (solutions) which satisfy it whereas a linear equation only has **one** solution.
We shall now describe three techniques for solving quadratic equations:

- factorisation
- completing the square
- using the quadratic formula

Exercises

1. Verify that $x = 2$ and $x = 3$ are both solutions of $x^2 - 5x + 6 = 0$.
2. Verify that $x = -2$ and $x = -3$ are both solutions of $x^2 + 5x + 6 = 0$.

2. Solution by factorisation

It may be possible to solve a quadratic equation by factorisation using the method described for factorising quadratic expressions in HELM 1.5, although you should be aware that not all quadratic equations can be easily factorised.

Example 10

Solve the equation $x^2 + 5x = 0$.

Solution

Factorising and equating each factor to zero we find

$$x^2 + 5x = 0 \quad \text{is equivalent to} \quad x(x + 5) = 0$$

so that $x = 0$ and $x = -5$ are the two solutions.

Example 11

Solve the quadratic equation $x^2 + x - 6 = 0$.

Solution

Factorising the left hand side we find $x^2 + x - 6 = (x + 3)(x - 2)$ so that

$$x^2 + x - 6 = 0 \quad \text{is equivalent to} \quad (x + 3)(x - 2) = 0$$

When the product of two quantities equals zero, at least one of the two must equal zero. In this case either $(x + 3)$ is zero or $(x - 2)$ is zero. It follows that

$$x + 3 = 0, \qquad \text{giving} \qquad x = -3 \quad \text{or} \quad x - 2 = 0, \qquad \text{giving} \qquad x = 2$$

Here there are two solutions, $x = -3$ and $x = 2$.

These solutions can be checked quite easily by substitution back into the given equation.

Example 12

Solve the quadratic equation $2x^2 - 7x - 4 = 0$ by factorising the left-hand side.

Solution

Factorising the left hand side: $2x^2-7x-4 = (2x+1)(x-4)$ so $2x^2-7x-4 = 0$ is equivalent to $(2x+1)(x-4) = 0$. In this case either $(2x+1)$ is zero or $(x-4)$ is zero. It follows that $2x+1 = 0$, giving $x = -\frac{1}{2}$ or $x - 4 = 0$, giving $x = 4$

There are two solutions, $x = -\frac{1}{2}$ and $x = 4$.

Example 13

Solve the equation $4x^2 + 12x + 9 = 0$.

Solution

Factorising we find $4x^2 + 12x + 9 = (2x+3)(2x+3) = (2x+3)^2$

This time the factor $(2x+3)$ occurs twice. The original equation $4x^2 + 12x + 9 = 0$ becomes

$$(2x+3)^2 = 0 \quad \text{so that} \quad 2x+3 = 0$$

and we obtain the solution $x = -\frac{3}{2}$. Because the factor $2x+3$ appears twice in the equation $(2x+3)^2 = 0$ we say that this root is a **repeated solution** or **double root**.

Task

Solve the quadratic equation $7x^2 - 20x - 3 = 0$.

First factorise the left-hand side:

Your solution

$7x^2 - 20x - 3 =$

Answer

$(7x+1)(x-3)$

Equate each factor is then equated to zero to obtain the two solutions:

Your solution

Solution 1: $x =$ Solution 2: $x =$

Answer

$-\frac{1}{7}$ and 3

Exercises

Solve the following equations by factorisation:

1. $x^2 - 3x + 2 = 0$	2. $x^2 - x - 2 = 0$	3. $x^2 + x - 2 = 0$
4. $x^2 + 3x + 2 = 0$	5. $x^2 + 8x + 7 = 0$	6. $x^2 - 7x + 12 = 0$
7. $x^2 - x - 20 = 0$	8. $4x^2 - 4 = 0$	9. $-x^2 + 2x - 1 = 0$
10. $3x^2 + 6x + 3 = 0$	11. $x^2 + 11x = 0$	12. $2x^2 + 2x = 0$

Answers The factors are found to be:

1. $1, 2$	2. $-1, 2$	3. $-2, 1$	4. $-1, -2$	5. $-7, -1$
6. $4, 3$	7. $-4, 5$	8. $1, -1$	9. 1 twice	10. -1 twice
11. $-11, 0$	12. $0, -1$			

3. Completing the square

The technique known as completing the square can be used to solve quadratic equations although it is applicable in many other circumstances too so it is well worth studying.

Example 14

(a) Show that $(x + 3)^2 = x^2 + 6x + 9$

(b) Hence show that $x^2 + 6x$ can be written as $(x + 3)^2 - 9$.

Solution

(a) Removing the brackets we find

$$(x + 3)^2 = (x + 3)(x + 3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9$$

(b) By subtracting 9 from both sides of the previous equation it follows that

$$(x + 3)^2 - 9 = x^2 + 6x$$

Example 15

(a) Show that $(x-4)^2 = x^2 - 8x + 16$

(b) Hence show that $x^2 - 8x$ can be written as $(x-4)^2 - 16$.

Solution

(a) Removing the brackets we find

$$(x-4)^2 = (x-4)(x-4) = x^2 - 4x - 4x + 16 = x^2 - 8x + 16$$

(b) Subtracting 16 from both sides we can write

$$(x-4)^2 - 16 = x^2 - 8x$$

We shall now generalise the results of Examples 14 and 15. Noting that

$(x+k)^2 = x^2 + 2kx + k^2$ we can write $x^2 + 2kx = (x+k)^2 - k^2$

Note that the constant term in the brackets on the right-hand side is always half the coefficient of x on the left. This process is called **completing the square**.

Key Point 4

Completing the Square

The expression $x^2 + 2kx$ is equivalent to $(x+k)^2 - k^2$

Example 16

Complete the square for the expression $x^2 + 16x$.

Solution

Comparing $x^2 + 16x$ with the general form $x^2 + 2kx$ we see that $k = 8$. Hence

$$x^2 + 16x = (x+8)^2 - 8^2 = (x+8)^2 - 64$$

Note that the constant term in the brackets on the right, that is 8, is half the coefficient of x on the left, which is 16.

Example 17

Complete the square for the expression $5x^2 + 4x$.

Solution

Consider $5x^2 + 4x$. First of all the coefficient 5 is removed outside a bracket as follows

$$5x^2 + 4x = 5(x^2 + \frac{4}{5}x)$$

We can now complete the square for the quadratic expression in the brackets:

$$x^2 + \frac{4}{5}x = (x + \frac{2}{5})^2 - \left(\frac{2}{5}\right)^2 = (x + \frac{2}{5})^2 - \frac{4}{25}$$

Finally, multiplying both sides by 5 we find

$$5x^2 + 4x = 5\left((x + \frac{2}{5})^2 - \frac{4}{25}\right)$$

Completing the square can be used to solve quadratic equations as shown in the following Examples.

Example 18

Solve the equation $x^2 + 6x + 2 = 0$ by completing the square.

Solution

First of all just consider $x^2 + 6x$, and note that we can write this as

$$x^2 + 6x = (x + 3)^2 - 9$$

Then the quadratic equation can be written as

$$x^2 + 6x + 2 = (x + 3)^2 - 9 + 2 = 0 \quad \text{that is} \qquad (x + 3)^2 = 7$$

Taking the square root of both sides gives

$$x + 3 = \pm\sqrt{7} \quad \text{so} \quad x = -3 \pm \sqrt{7}$$

The two solutions are $x = -3 + \sqrt{7} = -0.3542$ and $x = -3 - \sqrt{7} = -5.6458$, to 4 d.p.

Example 19

Solve the equation $x^2 - 8x + 5 = 0$

Solution

First consider $x^2 - 8x$ which we can write as $x^2 - 8x = (x-4)^2 - 16$ so that the equation becomes

$$x^2 - 8x + 5 = (x-4)^2 - 16 + 5 = 0$$

$$\text{i.e.}\qquad \begin{aligned}(x-4)^2 &= 11\\ x-4 &= \pm\sqrt{11}\\ x &= 4 \pm \sqrt{11}\end{aligned}$$

So $x = 7.3166$ or $x = 0.6834$ (to 4 d.p.)

Task

Solve the equation $x^2 - 4x + 1 = 0$ by completing the square.

First examine the two left-most terms in the equation: $x^2 - 4x$. Complete the square for these terms:

Your solution

$x^2 - 4x =$

Answer

$(x-2)^2 - 4$

Use the above result to rewrite the equation $x^2 - 4x + 1 = 0$ in the form $(x- \ ?)^2 + \ ? = 0$:

Your solution

$x^2 - 4x + 1 =$

Answer

$(x-2)^2 - 4 + 1 = (x-2)^2 - 3 = 0$

From this now obtain the roots:

Your solution

Answer

$(x-2)^2 = 3$, so $x - 2 = \pm\sqrt{3}$. Therefore $x = 2 \pm \sqrt{3}$ so $x = 3.7321$ or 0.2679 to 4 d.p.

Exercises

1. Solve the following quadratic equations by completing the square.

   (a) $x^2 - 3x = 0$

   (b) $x^2 + 9x = 0.$

   (c) $2x^2 - 5x + 2 = 0$

   (d) $6x^2 - x - 1 = 0$

   (e) $-5x^2 + 6x - 1 = 0$

   (f) $-x^2 + 4x - 3 = 0$

2. A chemical manufacturer found that the sales figures for a certain chemical X_2O depended on its selling price. At present, the company can sell all of its weekly production of 300 t at a price of £600 / t. The company's market research department advised that the amount sold would decrease by only 1 t per week for every £2 / t increase in the price of X_2O. If the total production costs are made up of a fixed cost of £30000 per week, plus £400 per t of product, show that the weekly profit is given by

$$P = -\frac{x^2}{2} + 800x - 270000$$

   where x is the new price per t of X_2O. Complete the square for the above expression and hence find

   (a) the price which maximises the weekly profit on sales of X_2O

   (b) the maximum weekly profit

   (c) the weekly production rate

Answers

1. (a) $0, 3$ (b) $0, -9$ (c) $2, \frac{1}{2}$ (d) $\frac{1}{2}, -\frac{1}{3}$ (e) $\frac{1}{5}, 1$ (f) $1, 3$

2. (a) £800 / t, (b) £50000 /wk, (c) 200 t / wk

4. Solution by formula

When it is difficult to factorise a quadratic equation, it may be possible to solve it using a formula which is used to calculate the roots. The formula is obtained by completing the square in the general quadratic $ax^2 + bx + c$. We proceed by removing the coefficient of a:

$$ax^2 + bx + c = a\{x^2 + \frac{b}{a}x + \frac{c}{a}\} = a\{(x + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}\}$$

Thus the solution of $ax^2 + bx + c = 0$ is the same as the solution to

$$(x + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

So, solving: $(x+\frac{b}{2a})^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$ which leads to $x = -\frac{b}{2a} \pm \sqrt{-\frac{c}{a} + \frac{b^2}{4a^2}}$

Simplifying this expression further we obtain the important result:

Key Point 5

Quadratic Formula

If $ax^2 + bx + c = 0, \quad a \neq 0$ then the two solutions (roots) are

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

To apply the formula to a specific quadratic equation it is necessary to identify carefully the values of a, b and c, paying particular attention to the signs of these numbers. Substitution of these values into the formula then gives the desired solutions.

Note that if the quantity $b^2 - 4ac$ (called the **discriminant**) is a positive number we can take its square root and the formula will produce two values known as **distinct real roots**. If $b^2 - 4ac = 0$ there will be one value only known as a **repeated root** or **double root**. The value of this root is $x = -\frac{b}{2a}$. Finally if $b^2 - 4ac$ is negative we say the equation possesses **complex roots**. These require special treatment and are described in HELM 10.

Key Point 6

When finding roots of the quadratic equation $ax^2 + bx + c = 0$ first calculate the discrinimant

$$b^2 - 4ac$$

- If $b^2 - 4ac > 0$ the quadratic has two real distinct roots
- If $b^2 - 4ac = 0$ the quadratic has two real and equal roots
- If $b^2 - 4ac < 0$ the quadratic has no real roots: there are two complex roots

Example 20

Compare each given equation with the standard form $ax^2+bx+c=0$ and identify a, b and c. Calculate b^2-4ac in each case and use this information to state the nature of the roots.

(a) $3x^2+2x-7=0$ (b) $3x^2+2x+7=0$

(c) $3x^2-2x+7=0$ (d) $x^2+x+2=0$

(e) $-x^2+3x-\frac{1}{2}=0$ (f) $5x^2-3=0$

(g) $x^2-2x+1=0$ (h) $2p^2-4p=0$

(i) $-p^2+4p-4=0$

Solution

(a) $a=3,\ b=2,\ c=-7$. So $b^2-4ac=(2)^2-4(3)(-7)=88$.
The roots are real and distinct.

(b) $a=3,\ b=2, c=7$. So $b^2-4ac=(2)^2-4(3)(7)=-80$.
The roots are complex.

(c) $a=3,\ b=-2,\ c=7$. So $b^2-4ac=(-2)^2-4(3)(7)=-80$.
The roots are complex.

(d) $a=1,\ b=1,\ c=2$. So $b^2-4ac=1^2-4(1)(2)=-7$.
The roots are complex.

(e) $a=-1,\ b=3,\ c=-\frac{1}{2}$. So $b^2-4ac=3^2-4(-1)(-\frac{1}{2})=7$.
The roots are real and distinct.

(f) $a=5,\ b=0,\ c=-3$. So $b^2-4ac=0-4(5)(-3)=60$.
The roots are real and distinct.

(g) $a=1,\ b=-2,\ c=1$. So $b^2-4ac=(-2)^2-4(1)(1)=0$.
The roots are real and equal.

(h) $a=2, b=-4, c=0$. So $b^2-4ac=(-4)^2-4(2)(0)=16$
The roots are real and distinct.

(i) $a=-1, b=4, c=-4$. So $b^2-4ac=(-4)^2-4(-1)(-4)=0$
The roots are real and equal.

Example 21

Solve the quadratic equation $2x^2 + 3x - 6 = 0$ using the formula.

Solution

We compare the given equation with the standard form $ax^2 + bx + c = 0$ in order to identify a, b and c. We see that here $a = 2$, $b = 3$ and $c = -6$. Note particularly the sign of c. Substituting these values into the formula we find

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{(2)(2)} = \frac{-3 \pm \sqrt{9+48}}{4} = \frac{-3 \pm 7.5498}{4}$$

Hence, to 4 d.p., the two roots are $x = 1.1375$, if the positive sign is taken and $x = -2.6375$ if the negative sign is taken. However, it is often sufficient to leave the solution in the so-called **surd form** $x = \dfrac{-3 \pm \sqrt{57}}{4}$, which is exact.

Task

Solve the equation $3x^2 - x - 6 = 0$ using the quadratic formula.

First identify a, b and c:

Your solution

$a =$ $\qquad$ $b =$ $\qquad$ $c =$

Answer

$a = 3, \qquad b = -1, \qquad c = -6$

Substitute these values into the formula and simplify:

Your solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{so} \quad x =$$

Answer

$$\frac{-(-1) \pm \sqrt{(-1)^2 - (4)(3)(-6)}}{(2)(3)} = \frac{1 \pm \sqrt{73}}{6}$$

Finally, calculate the values of x to 4 d.p.:

Your solution

$x =$ $\qquad$ or $\qquad$ $x =$

Answer

$1.5907, -1.2573$

Engineering Example 1

Undersea cable fault location

Introduction

The voltage (V), current (I) and resistance (R) in an electrical circuit are related by Ohm's law i.e. $V = IR$. If there are two resistances (R_1 and R_2) in an electrical circuit, they may be in series, in which case the total resistance (R) is given by $R = R_1 + R_2$. Or they may be in parallel in which case the total resistance is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

In 1871 the telephone cable between England (A) and Denmark (B) developed a fault, due to a short circuit under the sea (see Figure 2). Oliver Heaviside, an electrical engineer, came up with a very simple method to find the location of the fault. He assumed that the cable had a uniform resistance per unit length. Heaviside performed two tests:

(1) connecting a battery (voltage E) at A, with the circuit open at B, he measured the resulting current I_1,

(2) connecting the same battery at A, with the cable earthed at B, he measured the current I_2.

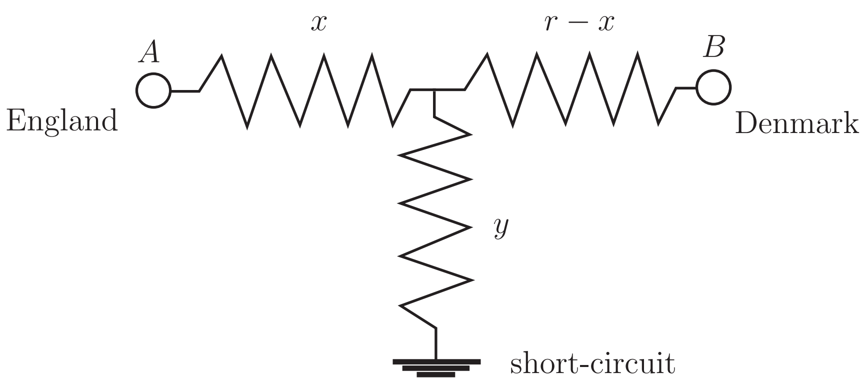


Figure 2: Schematic of the undersea cable

In the first measurement the resistances up to the cable fault and between the fault and the short circuit are in series and in the second experiment the resistances beyond the fault and between the fault and the short circuit are in parallel.

Problem in words

Use the information from the measurements to deduce the location of the fault.

Mathematical statement of problem

(a) Denote the resistances of the various branches by the symbols shown in Figure 2.

(b) Use Ohm's law to write down expressions that apply to each of the two measurements.

(c) Eliminate y from these expressions to obtain an expression for x.

Mathematical analysis

(a) In the first experiment the total circuit resistance is $x + y$. In the second experiment, the total circuit resistance is given by:

$$x + \left(\frac{1}{r - x} + \frac{1}{y}\right)^{-1}$$

So application of Ohm's law to each experimental situation gives:

$$E = I_1(x + y) \qquad (1)$$

$$E = I_2(x + \left(\frac{1}{r - x} + \frac{1}{y}\right)^{-1}) \qquad (2)$$

Rearrange Equation (1) to give $\dfrac{E}{I_1} - x = y$

Substitute for y in Equation (2), divide both sides by I_2 and introduce $\dfrac{E}{I_1} = r_1$ and $\dfrac{E}{I_2} = r_2$:

$$r_2 = (x + \left(\frac{1}{r - x} + \frac{1}{y}\right)^{-1})$$

Use a common denominator for the fractions on the right-hand side:

$$r_2 = (x + \left(\frac{(r - x)(r_1 - x)}{r_1 - x + r - x}\right)) = \frac{x(r_1 + r - 2x) + (r - x)(r_1 - x)}{(r_1 + r - 2x)}$$

Multiply through by $(r_1 + r - 2x)$:

$$r_2(r_1 + r - 2x) = x(r_1 + r - 2x) + (r - x)(r_1 - x)$$

Rearrange as a quadratic for x:

$$x^2 - 2r_2x - rr_1 + r_2r_1 + rr_2 = 0$$

Use the standard formula for solving quadratic equations

with $a = 1, b = -2r_2$ and $c = -rr_1 + r_2r_1 + rr_2$:

$$x = \frac{2r_2 \pm \sqrt{4r_2^2 - 4(-rr_1 + r_2r_1 + rr_2)}}{2} = r_2 \pm \sqrt{(r - r_2)(r_1 - r_2)}$$

Only positive solutions would be of interest.

Engineering Example 2

Estimating the mass of a pipe

Introduction

Sometimes engineers have to estimate component weights from dimensions and material properties. On some occasions, engineers prefer use of approximate formulae to exact ones as long as they are sufficiently accurate for the purpose. This Example introduces both of these aspects.

Problem in words

(a) Find the mass of a given length of pipe in terms of its inner and outer diameters and the density of the pipe material.

(b) Find the wall thickness of the pipe if the inner diameter is 0.15 m, the density is 7900 kg m^{-3} and the mass per unit length of pipe is 40 kg m^{-1}.

(c) Find an approximate method for calculating the mass of a given length of a thin-walled pipe and calculate the maximum ratio of inner and outer diameters that give an error of less than 10% when using the approximate method.

Mathematical statement of problem

(a) Denote the length of the pipe by L m and inside and outside diameters by d_i m and d_o m, respectively and the density by ρ kg m^{-3}. Assume that the pipe is cylindrical so its cross section corresponds to the gap between concentric circles (this is called an annulus or annular region - see HELM 2.6). Calculate the difference in cross sectional areas by using the formula for the area of a circle ($A = \pi r^2$ where r is the radius) and multiply by the density and length to obtain mass (m).

(b) Rearrange the equation in terms of wall thickness (d m) and inner diameter. Substitute the given values to determine the wall thickness.

(c) Approximate the resulting expression for small values of $(d_o - d_i)$. Calculate the percentage difference in predictions between the original and approximate formulae for various numerical values of d_i/d_o.

Mathematical analysis

(a) The cross section of a cylindrical pipe is a circular annulus. The area of a circle is given by $\pi r^2 = \dfrac{\pi}{4}d^2$, since $r = d/2$ if d is the diameter. So the area of the outer circle is $\dfrac{\pi}{4}d_o^2$ and that of the inner circle is $\dfrac{\pi}{4}d_i^2$. This means that the mass m kg of length L m of the pipe is given by

$$m = \frac{\pi}{4}(d_0^2 - d_i^2)L\rho$$

(b) Denote the pipe wall thickness by δ so $d_o = d_i + 25$.

Use $(d_o^2 - d_i^2) = (d_o - d_i)(d_o + d_i) = 2\delta(2d_i + 2\delta)$. So $\quad m = \pi\delta(d_i + \delta)L\rho$

Given that $m/L = 40,\ d_i = 0.15$ and $r = 7900$,

$$40 = pd(0.15 + d)7900.$$

Rearrange this equation as a quadratic in δ,

$$\delta^2 + 0.15\delta - 4\pi/790 = 0$$

Solve this quadratic using the standard formula with $a = 1, b = 0.15$ and $c = 4\pi/790$. Retain only the positive solution to give $\delta = 0.072$, i.e. the pipe wall thickness is 72 mm.

(c) If δ is small then $(d_o - d_i)$ is small and $d_i + \delta \approx d_i$. So the expression for m in terms of δ may be written

$$m \approx \pi\delta d_i L\rho$$

The graph in Figure 3 shows that the percentage error from using the approximate formula for the mass of the pipe exceeds 10% only if the inner diameter is less than 82% of the outer diameter.

The percentage error from using the approximate formula can be calculated from (exact result − approximate result)/(exact result) × 100% for various values of the ratio of inner to outer diameters. In the graph the error is plotted for diameter ratios between 0.75 and 1.

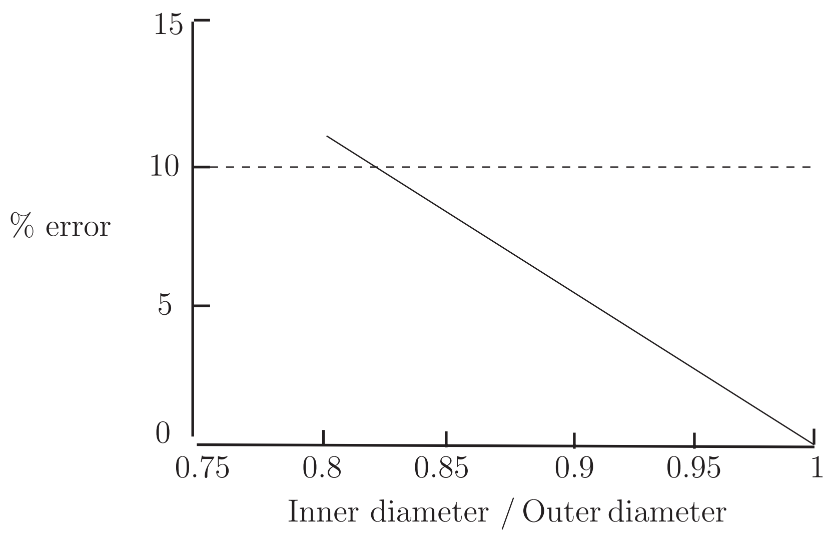


Figure 3

Comment

The graph shows also that the error is 1% or less for diameter ratios > 0.98.

Exercises

Solve the following quadratic equations by using the formula. Give answers exactly (where possible) or to 4 d.p.:

1. $x^2 + 8x + 1 = 0$	2. $x^2 + 7x - 2 = 0$	3. $x^2 + 6x - 2 = 0$
4. $-x^2 + 3x + 1 = 0$	5. $-2x^2 - 3x + 1 = 0$	6. $2x^2 + 5x - 3 = 0$

Answers

1. $-0.1270,\ -7.8730$	2. $-7.2749, 0.2749$	3. 0.3166, -6.3166
4. 3.3028, -0.3028	5. $-1.7808, 0.2808$	6. $\frac{1}{2}, -3$

5. Geometrical representation of quadratics

We can plot a graph of the function $y = ax^2 + bx + c$ (given the values of a, b and c). If the graph crosses the horizontal axis it will do so when $y = 0$, and so the x coordinates at such points are solutions of $ax^2 + bx + c = 0$. Depending on the sign of a and of the nature of the solutions there are essentially six different types of graph that can occur. These are displayed in Figure 4.

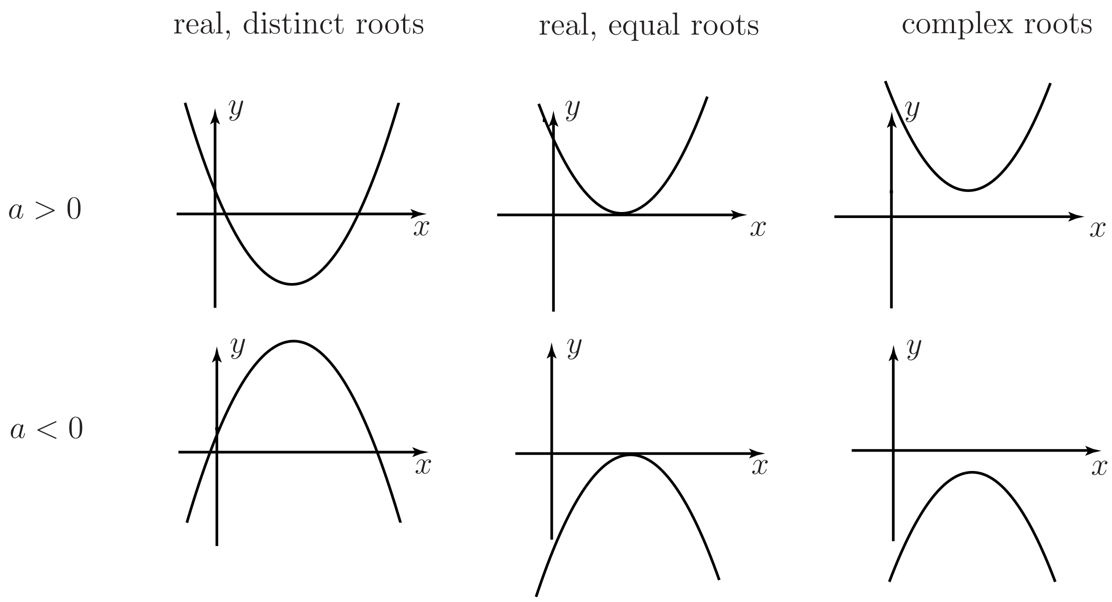


Figure 4: The possible graphs of a quadratic $y = ax^2 + bx + c$

Sometimes a graph of the quadratic is used to locate the solutions; however, this approach is generally inaccurate. This is illustrated in the following example.

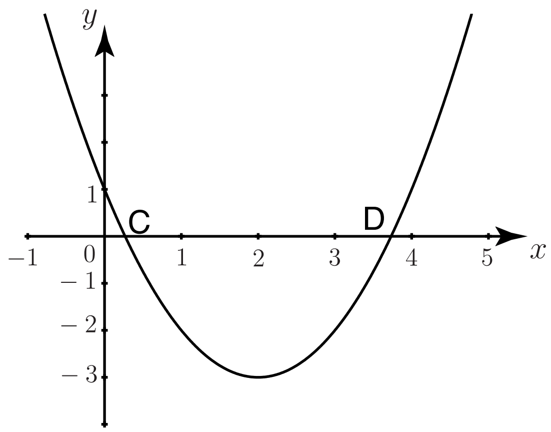
Example 22

Solve the equation $x^2 - 4x + 1 = 0$ by plotting a graph of the function:

$$y = x^2 - 4x + 1$$

Solution

By constructing a table of function values we can plot the graph as shown in Figure 5.



x	0	1	2	3	4
y	1	−2	−3	−2	1

Figure 5: The graph of $y = x^2 - 4x + 1$ cuts the x axis at C and D

The solutions of the equation $x^2 - 4x + 1 = 0$ are found by looking for points where the graph crosses the horizontal axis. The two points are approximately $x = 0.3$ and $x = 3.7$ marked C and D on the Figure.

Exercises

1. Solve the following quadratic equations giving exact numeric solutions. Use whichever method you prefer

   (a) $x^2 - 9 = 0$ (b) $s^2 - 25 = 0$
   (c) $3x^2 - 12 = 0$ (d) $x^2 - 5x + 6 = 0$
   (e) $6s^2 + s - 15 = 0$ (f) $p^2 + 7p = 0$

2. Solve the equation $2x^2 - 3x - 7 = 0$ giving solutions rounded to 4 d.p.

3. Solve the equation $2t^2 + 3t - 4$ giving the solutions in surd form.

Answers

1(a) $x = 3, -3$, (b) $s = 5, -5$, (c) $x = 2, -2$, (d) $x = 3, 2$, (e) $s = 3/2, -5/3$,

(f) $p = 0, -7$. 2. $-2.7656, 1.2656$. 3. $\dfrac{-3 \pm \sqrt{43}}{4}$