The Circle

➧ Introduction

A circle is one of the most familiar geometrical figures and has been around a long time! In this brief Section we discuss the basic coordinate geometry of a circle - in particular the basic equation representing a circle in terms of its centre and radius.

➧ **Prerequisites**	• understand what is meant by a function and be able to use functional notation
Before starting this Section you should . . .	• be able to plot graphs of functions

➧ **Learning Outcomes**	• obtain the equation of any given circle
On completion you should be able to . . .	• obtain the centre and radius of a circle from its equation

1. Equations for circles in the Oxy plane

The obvious characteristic of a circle is that every point on its circumference is the **same** distance from the **centre**. This fixed distance is called the **radius** of the circle and is generally denoted by R or r or a.

In coordinate geometry terms suppose (x, y) denotes the coordinates of a point. For example, (4,2) means $x = 4, \ y = 2$, $(-1, 1)$ means $x = -1, \ y = 1$ and so on. See Figure 26.
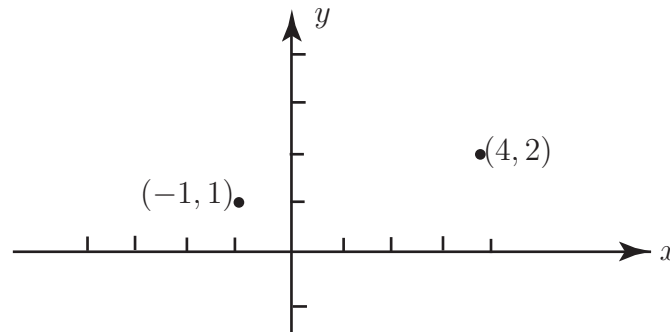
Figure 26

Example 14

Write down the distances d_1 and d_2 from the origin of the points with coordinates (4,2) and $(-1, 1)$ respectively. Generalise the result to obtain the distance d from the origin of **any** arbitrary point with coordinates (x, y).

Solution

Using Pythagoras' Theorem:

$d_1 = \sqrt{4^2 + 2^2} = \sqrt{20}$ is the distance between the origin (0,0) and the point (4,2).

$d_2 = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ is the distance between the origin and $(-1, 1)$.

$d = \sqrt{x^2 + y^2}$ is the distance from the origin to an arbitrary point (x, y). Note that the positive square root is taken in each case.

Circles with centre at the origin

Suppose (x, y) is any point P on a circle of radius R whose centre is at the origin. See Figure 27.
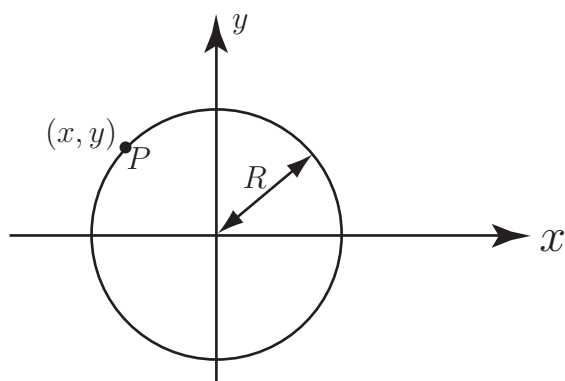

Figure 27

> **Task**
>
> Using the final result of Example (14), write down an equation relating x, y and R.

Your solution

Answer

Since $\sqrt{x^2 + y^2}$ is distance of any point (x, y) from the origin, then for any point P on the above circle.

$$\sqrt{x^2 + y^2} = R \qquad \text{or} \qquad x^2 + y^2 = R^2$$

As the point P in Figure 27 moves around the circle its x and y coordinates change. However P will remain at the same distance R from the origin by the very definition of a circle.

Hence we say that

$$\sqrt{x^2 + y^2} = R \quad \text{or, more usually,}$$

$$x^2 + y^2 = R^2 \tag{1}$$

is **the equation** of the circle radius R centre at the origin. What this means is that if a point (x, y) satisfies (1) then it lies on the circumference of the circle radius R. If (x, y) does not satisfy (1) then it does not lie on that circumference.

Note carefully that the right-hand sides of the circle equation (1) is the **square** of the radius.

Task

Consider the circle centre at the origin and of radius 5.

> (a) Write down the equation of this circle.
>
> (b) For the following points determine which lie on the circumference of this circle, which lie inside the circle and which lie outside the circle.
>
> $(5, 0)$ $(0, -5)$ $(4, 3)$ $(-3, 4)$ $(2, \sqrt{21})$ $(-2\sqrt{6}, 1)$ $(1, 4)$ $(4, -4)$
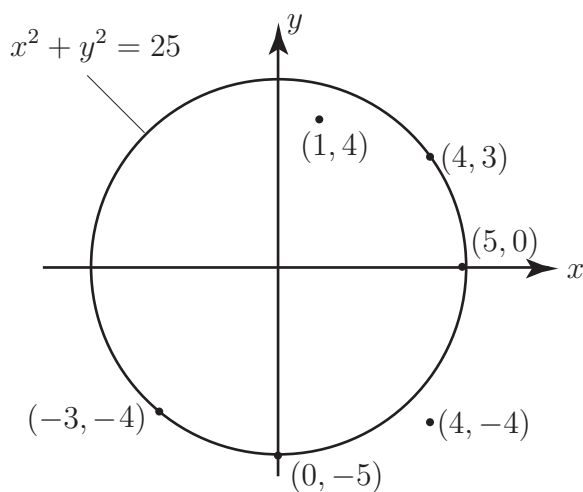
Your solution

(a)

(b)

(x, y)	$x^2 + y^2$	conclusion
$(5, 0)$		
$(0, -5)$		
$(4, 3)$		
$(-3, -4)$		
$(2, \sqrt{21})$		
$(-2\sqrt{6}, 1)$		
$(1, 4)$		
$(4, -4)$		

Answer

(a) $x^2 + y^2 = 5^2 = 25$ is the equation of the circle.

(b) For each point (x, y) we calculate $x^2 + y^2$. If this equals 25 the point lies on the circle' if greater than 25 then outside and if less than 25 then inside.

x, y	$x^2 + y^2$	conclusion
$(5, 0)$	25	on circle
$(0, -5)$	25	on circle
$(4, 3)$	25	on circle
$(-3, -4)$	25	on circle
$(2, \sqrt{21})$	25	on circle
$(-2\sqrt{6}, 1)$	25	on circle
$(1, 4)$	17	inside circle
$(4, -4)$	32	outside circle

Figure 28 demonstrates some of the results of the previous Task.

$$x^2 + y^2 = 25$$
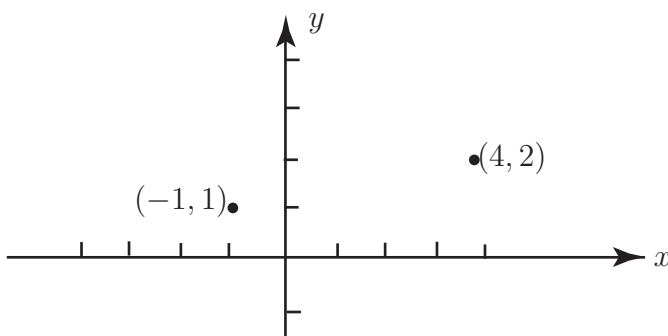
(1, 4) (4, 3)

(5, 0)

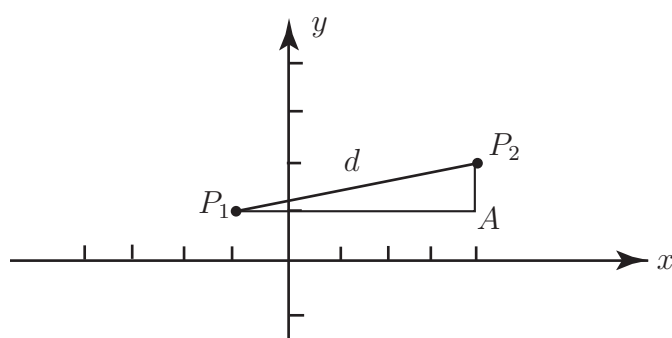(−3, −4) (4, −4)

(0, −5)

Figure 28

Note that the circle centre at the origin and of radius 1 has a special name – the **unit circle.**

Task

Calculate the distance between the points $P_1(-1, 1)$ and $P_2(4, 2)$.

(−1, 1)●   ●(4, 2)

Your solution

Answer

Using Pythagoras' Theorem the distance between the two given points is

$$d = \sqrt{(P_1 A)^2 + (AP_2)^2}$$
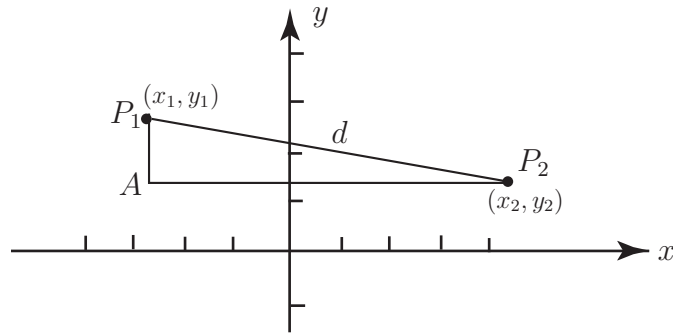
where $P_1 A = 4 - (-1) = 5, \quad AP_2 = 2 - 1 = 1$

$$\therefore \quad d = \sqrt{5^2 + 1^2} = \sqrt{26}$$

Task

Generalise your result to the previous Task to obtain the distance between any two points whose coordinates are (x_1, y_1) and (x_2, y_2).
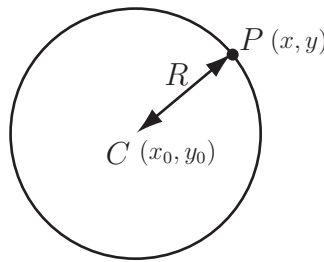
Your solution

Answer

Between the arbitrary points P_1 and P_2 the distance is

$$d = \sqrt{(AP_2)^2 + (P_1 A)^2}$$

where $AP_2 = x_2 - x_1$, $P_1 A = y_1 - y_2 = -(y_2 - y_1)$

so $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Using the result of the last Task, we now consider a circle centre at the point $C(x_0; y_0)$ and of radius R. Suppose P is an arbitrary point on this circle which has co-ordinates (x, y):

Figure 29

Clearly $R = CP = \sqrt{(x - x_0)^2 + (y - y_0)^2}$
Hence, squaring both sides,

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \tag{2}$$

which is said to be the equation of the circle centre (x_0, y_0) radius R.

Note that if $x_0 = y_0 = 0$ (i.e. circle centre is at origin) then (2) reduces to (1) so the latter is simply a special case.

The interpretation of (2) is similar to that of (1): any point (x, y) satisfying (2) lies on the circumference of the circle.

Example 15

What does the equation $(x - 3)^2 + (y - 4)^2 = 4$ represent?

Solution

It represents a circle of radius 2 (the positive square root of 4) and has centre $C(3, 4)$.

N.B. There is no need to expand the terms on the left-hand side of the equation here. The given form reveals quite plainly the radius ($\sqrt{4}$) and centre $(3, 4)$ of the circle.

Task

Write down the equations of each of the following circles for which the centre C and radius R are given:

(a) $C(0, 2), \; R = 2$

(b) $C(-2, 0), \; R = 3$

(c) $C(-3, 4), \; R = 5$

(d) $C(1, 1), \; R = \sqrt{3}$

Your solution

(a)

(b)

(c)

(d)

Answer

(a) $x_0 = 0, \; y_0 = 2, \; R^2 = 4$ so by Equation (2) the circle's equation is
$$x^2 + (y - 2)^2 = 4$$

(b) $x_0 = -2, \; y_0 = 0, \; R^2 = 9 \qquad \therefore \qquad (x + 2)^2 + y^2 = 9$
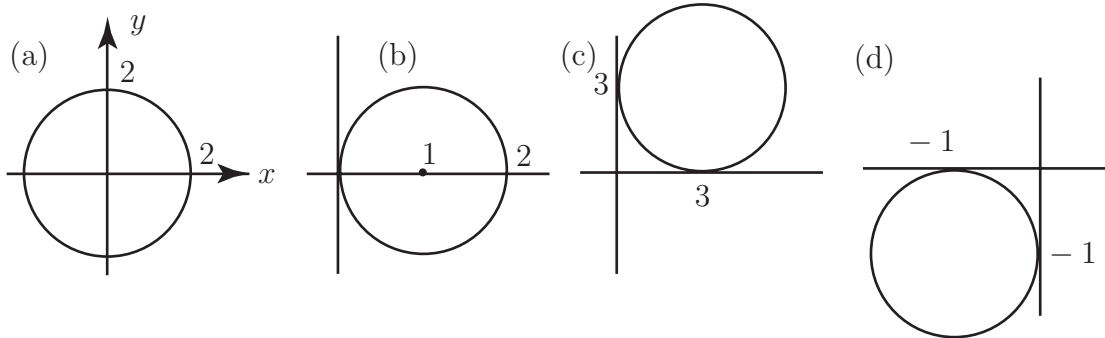
(c) $(x + 3)^2 + (y - 4)^2 = 25$

(d) $(x - 1)^2 + (y - 1)^2 = 3$

Again we emphasise that the right-hand side of each of these equations is the square of the radius.

Task

Write down the equations of each of the circles shown below:

(a)

(b)

(c)

(d)

Your solution

Answer

    (a) $x^2 + y^2 = 4$ (centre $(0,0)$ i.e. the origin, radius 2)

    (b) $(x-1)^2 + y^2 = 1$ (centre $(1,0)$, radius 1)

    (c) $(x-3)^2 + (y-3)^2 = 0$ (centre $(3,3)$, radius 3)

    (d) $(x+1)^2 + (y+1)^2 = 1$ (centre $(-1,-1)$, radius 1)

Consider again the equation of the circle, centre (3,4) of radius 2:

$$(x-3)^2 + (y-4)^2 = 4 \tag{3}$$

In this form of the equation the centre and radius of the circle can be clearly identified and, as we said, there is no advantage in squaring out. However, if we did square out the equation would become

$$x^2 - 6x + 9 + y^2 - 8x + 16 = 4 \quad \text{or} \quad x^2 - 6x + y^2 - 8x + 21 = 0 \tag{4}$$

Equation (4) is of course a valid equation for this circle but, we cannot immediately obtain the centre and radius from it.

Task

For the case of the general circle of radius R

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

expand out the square terms and simplify.

Your solution

Answer
We obtain

$$x^2 - 2x_0 x + x_0^2 + y^2 - 2y_0 y + y_0^2 - R^2 = 0$$

or

$$x^2 + y^2 - 2x_0 x - 2y_0 y + c = 0$$

where the constant $c = x_0^2 + y_0^2 - R^2$.

It follows from the above task that any equation of the form

$$x^2 + y^2 - 2gx - 2fy + c = 0 \tag{5}$$

represents a circle with centre (g, f) and a radius obtained by solving

$$c = g^2 + f^2 - R^2$$

for R.
Thus

$$R = \sqrt{g^2 + f^2 - c} \tag{6}$$

There is no need to remember Equation (6). In any specific problem the technique of **completion of square** can be used to turn an equation of the form (5) into the form of Equation (2) (i.e. $(x - x_0)^2 + (y - y_0)^2 = R^2$) and hence obtain the centre and radius of the circle.

NB. The key point about Equation (5) is that the coefficients of the term x^2 and y^2 are the **same**, i.e. 1. An equation with the coefficient of x^2 and y^2 identical with value $k \neq 1$ could be converted into the form (5) by division of the whole equation by k.

Task

If
$$x^2 + y^2 - 2x + 10y + 16 = 0$$

obtain the centre and radius of the circle that this equation represents.

Begin by completing the square separately on the $x-$terms and the $y-$terms:

Your solution

Answer

$$\begin{aligned} x^2 - 2x &= (x-1)^2 - 1 \\ y^2 + 10y &= (y+5)^2 - 25 \end{aligned}$$

Now complete the problem:

Your solution

Answer

The original equation

$$x^2 + y^2 - 2x + 10y + 16 = 0$$

becomes

$$(x-1)^2 - 1 + (y+5)^2 - 25 + 16 = 0$$

$$\therefore \quad (x-1)^2 + (y+5)^2 = 10$$

which represents a circle with centre $(1, -5)$ and radius $\sqrt{10}$.

Circles and functions

Let us return to the equation of the unit circle

$$x^2 + y^2 = 1$$

Solving for y we obtain

$$y = \pm\sqrt{1 - x^2}.$$

This equation does not represent a function because of the two possible square roots which imply that for any value of x there are **two** values of y. (You will recall from earlier in this Workbook that a function requires only **one** value of the dependent variable y corresponding to each value of the independent variable x.)

However two functions can be obtained in this case: $\quad y = y_1 = +\sqrt{1 - x^2} \qquad y = y_2 = -\sqrt{1 - x^2}$

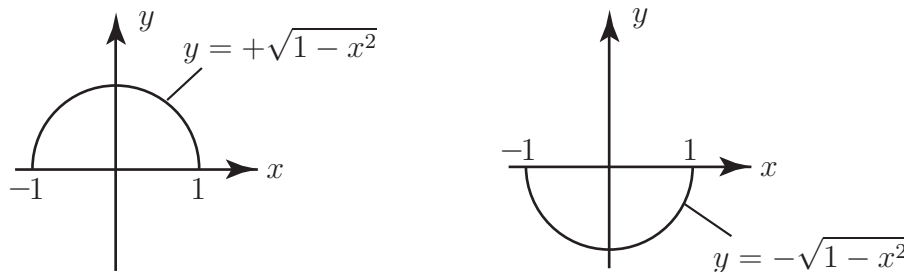whose graphs are the semicircles shown.

Figure 30

2. Annuli between circles

Equations in x and y, such as (1) i.e. $x^2 + y^2 = R^2$ and (2) i.e. $(x - x_0)^2 + (y - y_0)^2 = R^2$ for circles, define **curves** in the Oxy plane. However, inequalities are necessary to define **regions**. For example, the inequality

$$x^2 + y^2 < 1$$

is satisfied by all points **inside** the unit circle - for example $(0, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{4}, 0)$, $(\frac{1}{2}, \frac{1}{2})$.
Similarly $\quad x^2 + y^2 > 1\quad$ is satisfied by all points outside that circle such as $(1, 1)$.
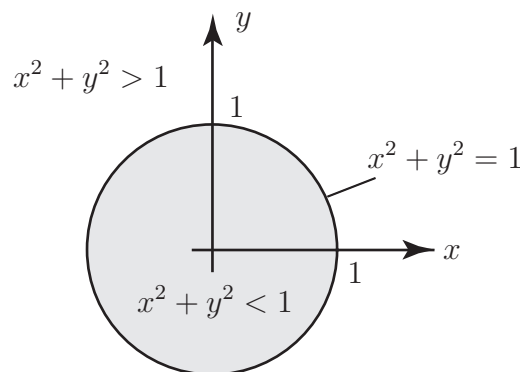

Figure 31

Example 16

Sketch the regions in the Oxy plane defined by

(a) $(x-1)^2 + y^2 < 1$ (b) $(x-1)^2 + y^2 > 1$

Solution

The equality $(x-1)^2 + y^2 = 1$ is satisfied by any point on the circumference of the circle centre $(1,0)$ radius 1. Then, remembering that $(x-1)^2 + y^2$ is the square of the distance between any point (x,y) and $(1,0)$, it follows that

    (a) $(x-1)^2 + y^2 < 1$ is satisfied by any point inside this circle (region (A) in the diagram.)

    (b) $(x-1)^2 + y^2 > 1$ defines the region exterior to the circle since this inequality is satisfied by every point outside. (Region (B) on the diagram.)
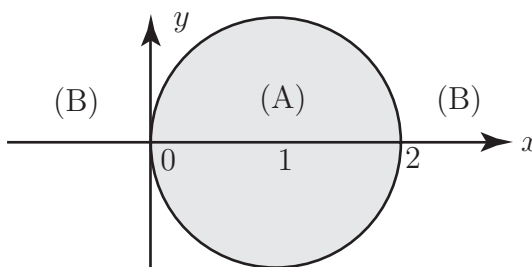
The region between two circles with the same centre (i.e. **concentric** circles) is called an **annulus** or **annular region**. An annulus is defined by **two** inequalities. For example the inequality

$$x^2 + y^2 > 1 \tag{7}$$

defines, as we saw, the region outside the unit circle.
The inequality

$$x^2 + y^2 < 4 \tag{8}$$

defines the region inside the circle centre origin radius 2.
Hence points (x,y) which satisfy **both** the inequalities (7) and (8) lie in the annulus between the two circles. The inequalities (7) and (8) are combined by writing
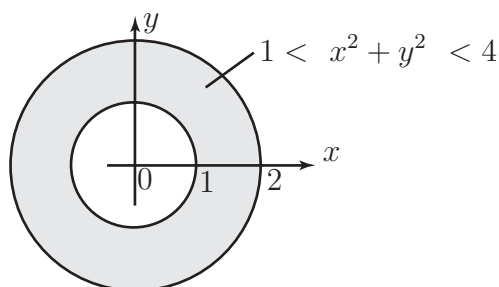
$$1 < x^2 + y^2 < 4$$

Figure 32

Task

Sketch the annulus defined by the inequalities

$$1 < (x - 1)^2 + y^2 < 9$$

Your solution

Answer

The quantity $(x - 1)^2 + y^2$ is the square of the distance of a point (x, y) from the point $(1,0)$. Hence, as we saw earlier, the left-hand inequality
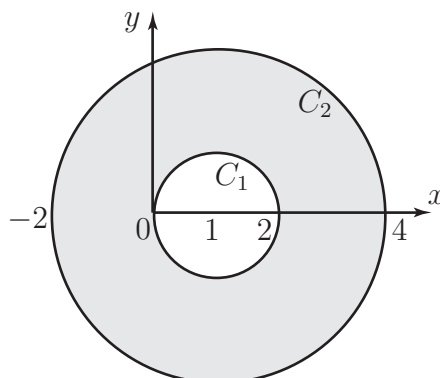
$$1 < (x - 1)^2 + y^2 \qquad \text{which is the same as} \qquad (x - 1)^2 + y^2 > 1$$

is the region exterior to the circle C_1 centre $(1, 0)$ radius 1.

Similarly the right-hand inequality

$$(x - 1)^2 + y^2 < 9$$

defines the interior of the circle C_2 centre $(1, 0)$ radius 3. Hence the double inequality holds for any point in the annulus between C_1 and C_2.

Exercises

1. Write down the radius and the coordinates of the centre of the circle for each of the following equations

   (a) $x^2 + y^2 = 16$

   (b) $(x - 4)^2 + (y - 3)^2 = 12$

   (c) $(x + 3)^2 + (y - 1)^2 = 25$

   (d) $x^2 + (y + 1)^2 - 4 = 0$

   (e) $(x + 6)^2 + y^2 - 36 = 0$

2. Obtain in each case the equation of the given circle

   (a) centre C $(0, 0)$ radius 7

   (b) centre C $(0, 2)$ radius 2

   (c) centre C $(4, -4)$ radius 4

   (d) centre C $(-2, -2)$ radius 4

   (e) centre C $(-6, 0)$ radius 5

3. Obtain the radius and the coordinates of the centre for each of the following circles

   (a) $x^2 + y^2 - 10x + 12y = 0$

   (b) $x^2 + y^2 + 2x - 4y = 11$

   (c) $x^2 + y^2 - 6x - 16 = 0$

4. Describe the regions defined by each of these inequalities

   (a) $x^2 + y^2 > 4$

   (b) $x^2 + y^2 < 16$

   (c) the inequalities in (i) and (ii) together

5. State an inequality that describes the points that lie outside the circle of radius 4 with centre $(-4, 2)$.

6. State an inequality that describes the points that lie inside the circle of radius $\sqrt{6}$ with centre $(-2, -1)$.

7. Obtain the equation of the circle which has centre $(3, 4)$ and which passes through the point $(0, 5)$.

8. Show that if $A(x_1, y_1)$ and $B(x_2, y_2)$ are at opposite ends of a diameter of a circle then the equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

   (Hint: if P is any point on the circle obtain the slopes of the lines AP and BP and recall that the angle in a semicircle must be a right-angle.)

9. State the equation of the unique circle which **touches** the $x-$axis at the point $(2,0)$ and which passes through the point $(-1, 9)$.

Answers

1. (a) radius 4 centre $(0,0)$

   (b) radius $\sqrt{12}$ centre $(4,3)$

   (c) radius 5 centre $(-3,1)$

   (d) radius 2 centre $(0,-1)$

   (e) radius 6 centre $(-6,0)$

2. (a) $x^2 + y^2 = 49$

   (b) $x^2 + (y-2)^2 = 4$

   (c) $(x-4)^2 + (y+4)^2 = 16$

   (d) $(x+2)^2 + (y+2)^2 = 16$

   (e) $(x+6)^2 + y^2 = 25$

3. (a) centre $(5,-6)$ radius $\sqrt{61}$

   (b) centre $(-1,2)$ radius 4

   (c) centre $(3,0)$ radius 5

4. (a) the region outside the circumference of the circle centre the origin radius 2.

   (b) the region inside the circle centre the origin radius 4 (often referred to as a circular disc)

   (c) the annular ring between these two circles.

5. $(x+4)^2 + (y-2)^2 > 16$

6. $(x+2)^2 + (y+1)^2 < 6$

7. $(x-3)^2 + (y-4)^2 = 10$

8. $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$.

9. $(x-2)^2 + (y-5)^2 = 25$ (Note: since we are told the circle touches the $x-$axis at (2,0) the centre of the circle must be at the point $(2, y_0)$ where $y_0 = R$).