One-to-One and Inverse Functions

2.3

Introduction

In this Section we examine more terminology associated with functions. We explain one-to-one and many-to-one functions and show how the rule associated with certain functions can be reversed to give so-called inverse functions. These ideas will be needed when we deal with particular functions in later Sections.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a function
- be able to sketch graphs of simple functions

Learning Outcomes

On completion you should be able to ...

- explain what is meant by a one-to-one function
- explain what is meant by a many-to-one function
- explain what is meant by an inverse function, and determine when and how such a function can be found

1. One-to-many rules, many-to-one and one-to-one functions

One-to-many rules

Recall from Section 2.1 that a rule for a function must produce a single output for a given input. Not all rules satisfy this criterion. For example, the rule 'take the square root of the input' cannot be a rule for a function because for a given input there are two outputs; an input of 4 produces outputs of 2 and -2. Figure 10 shows two ways in which we can picture this situation, the first being a block diagram, and the second using two sets representing input and output values and the relationship between them.

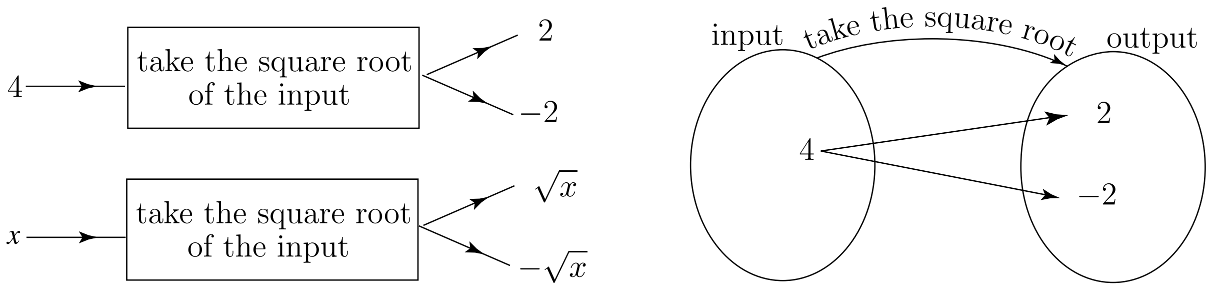

Figure 10: This rule cannot be a function - it is a one-to-many rule

Such a rule is described as a **one-to-many rule**. This means that one input produces more than one output. This is obvious from inspecting the sets in Figure 10.
The graph of the rule 'take $\pm\sqrt{x}$' can be drawn by constructing a table of values:

Table 4

x	0	1	2	3	4
$y = \pm\sqrt{x}$	0	± 1	$\pm\sqrt{2}$	$\pm\sqrt{3}$	± 2

The graph is shown in Figure 11(a). For each value of x there are two corresponding values of y. Plotting a graph of a one-to-many rule will result in a curve through which a vertical line can be drawn which cuts the curve more than once as you can see. The vertical line cuts the curve more than once because there is more than one y value for each x value.

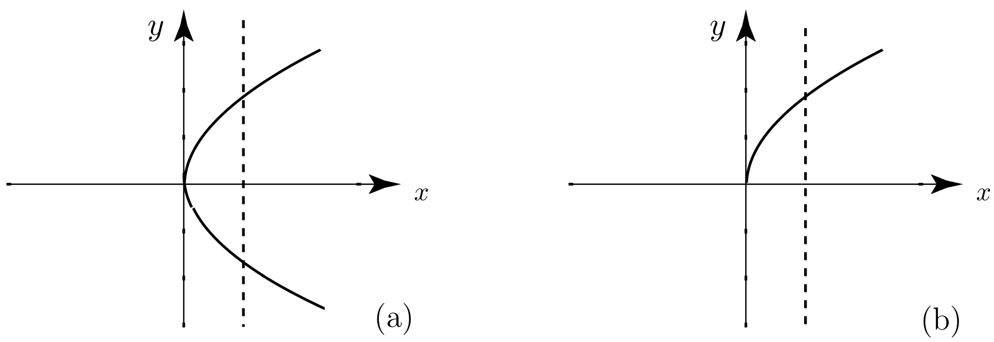

Figure 11

By describing a rule more carefully it is possible to make sure a single output results from a single input, thereby defining a valid rule for a function. For example, the rule 'take the **positive** square root of the input' is a valid function rule because a given input produces a single output. The graph of this function is displayed in Figure 11(b).

Many-to-one and one-to-one functions

Consider the function $y(x) = x^2$. An input of $x = 3$ produces an output of 9. Similarly, an input of -3 also produces an output of 9. In general, a function for which different inputs can produce the same output is called a **many-to-one function**. This is represented pictorially in Figure 12 from which it is clear why we call this a many-to-one function.

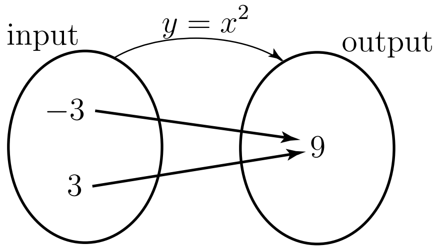

Figure 12: This represents a many-to-one function

Note that whilst this is many-to-one it is still a function since any chosen input value has only one arrow emerging from it. Thus there is a single output for each input.
It is possible to decide if a function is many-to-one by examining its graph. Consider the graph of $y = x^2$ shown in Figure 13.

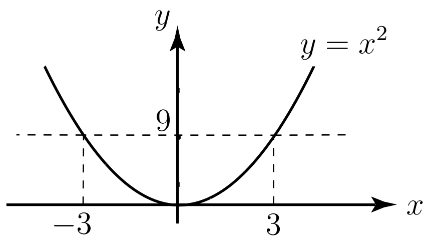

Figure 13: The function $y = x^2$ is a many-to-one function

We see that a horizontal line drawn on the graph cuts it more than once. This means that two (or more) different inputs have yielded the same output and so the function is many-to-one.
If a function is not many-to-one then it is said to be **one-to-one**. This means that each different input to the function yields a different output.

Consider the function $y(x) = x^3$ which is shown in Figure 14. A horizontal line drawn on this graph will intersect the curve only once. This means that each input value of x yields a different output value for y.

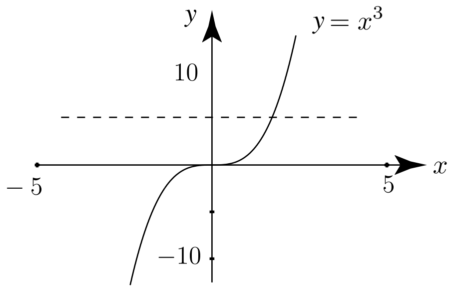

Figure 14: The function $y(x) = x^3$ is a one-to-one function

Study the graphs shown in Figure 15. Decide which, if any, are graphs of functions. For those which are, state if the function is one-to-one or many-to-one.

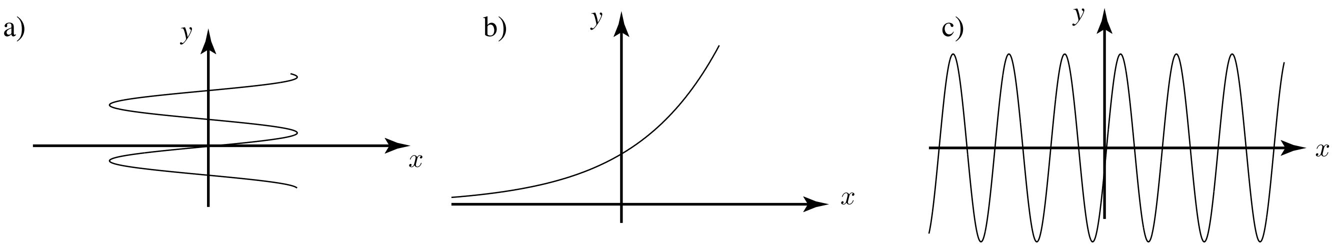

Figure 15

Your solution

Answer

(a) not a function, (b) one-to-one function, (c) many-to-one function

2. Inverse of a function

We have seen that a function can be regarded as taking an input, x, and processing it in some way to produce a single output $f(x)$ as shown in Figure 16(a). A natural question to ask is whether we can find another function that will reverse the process. In other words, can we find a function that will start with $f(x)$ and process it to produce x again? This idea is also shown in Figure 16(b). If we can find such a function it is called the **inverse function** to $f(x)$ and is given the symbol $f^{-1}(x)$. Do not confuse the '-1' with an index, or power. Here the superscript is used purely as the notation for the inverse function. Note that the composite function $f^{-1}(f(x)) = x$ as shown in Figure 17.

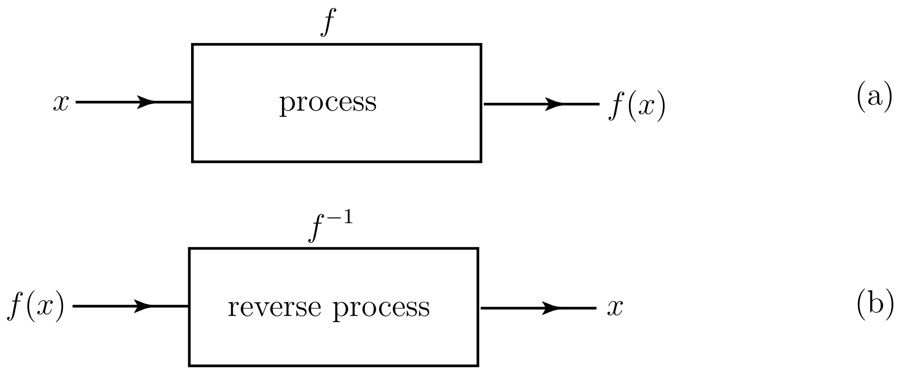

Figure 16: The second block reverse the process in the first

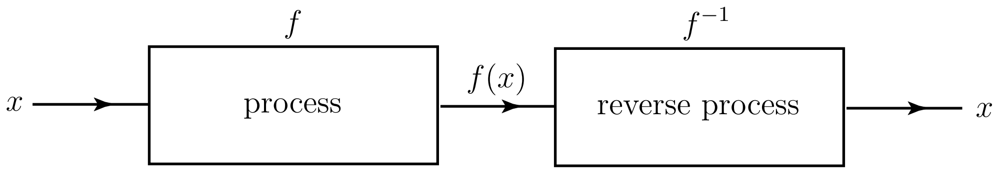

Figure 17: f^{-1} reverses the process in f

Example 6

Find the inverse function to $f(x) = 3x - 8$.

Solution

The given function takes an input, x and produces an output $3x - 8$. The inverse function, f^{-1}, must take an input $3x - 8$ and give an output x. That is

$$f^{-1}(3x - 8) = x$$

If we introduce a new variable $z = 3x - 8$, and transpose this for x to give

$$x = \frac{z+8}{3} \qquad \text{then} \qquad f^{-1}(z) = \frac{z+8}{3}$$

So the rule for f^{-1} is add 8 to the input and divide the result by 3. Writing f^{-1} with x as its argument gives

$$f^{-1}(x) = \frac{x+8}{3}$$

This is the inverse function.

Not all functions possess an inverse function. In fact, only one-to-one functions do so. If a function is many-to-one the process to reverse it would require many outputs from one input contradicting the definition of a function.

Task

Find the inverse of the function $f(x) = 7 - 3x$, using the fact that the inverse function must take an input $7-3x$ and produce an output x. So $f^{-1}(7-3x) = x$

Introduce a new variable z so that $z = 7 - 3x$ and transpose this to find x. Hence write down the inverse function:

Your solution

Answer

$f^{-1}(z) = \dfrac{7-z}{3}$. With x as its argument the inverse function is $f^{-1}(x) = \dfrac{7-x}{3}$.

Exercises

1. Explain why a one-to-many rule cannot be a function.
2. Illustrate why $y = x^4$ is a many-to-one function by providing a suitable example.
3. By sketching a graph of $y = 3x - 1$ show that this is a one-to-one function.
4. Explain why a many-to-one function does not have an inverse function. Give an example.
5. Find the inverse of each of the following functions:

(a) $f(x) = 4x + 7$, (b) $f(x) = x$, (c) $f(x) = -23x$, (d) $f(x) = \dfrac{1}{x+1}$.

Answers

5. (a) $f^{-1}(x) = \dfrac{x-7}{4}$, (b) $f^{-1}(x) = x$, (c) $f^{-1}(x) = -\dfrac{x}{23}$, (d) $f^{-1}(x) = \dfrac{1-x}{x}$.