

Basic Functions

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Learning outcomes

In this Workbook you will learn about some of the basic building blocks of mathematics. You will gain familiarity with functions and variables. You will learn how to graph a function and what is meant by an inverse function. You will learn how to use a parametric approach to describe a function. Finally, you will meet some of the functions which occur in engineering and science: polynomials, rational functions, the modulus function and the unit step function.

Basic Concepts of Functions

2.1



Introduction

In engineering there are many quantities that change their value as time changes. For example, the temperature of a furnace may change with time as it is heated. Similarly, there are many quantities that change their value as the location of a point of interest changes. For example, the shear stress in a bridge girder will vary from point to point across the bridge. A quantity whose value can change is known as a **variable**. We use **functions** to describe how one variable changes as a consequence of another variable changing. There are many different types of function that are used by engineers. We will be examining some of these in later Sections. The purpose of this Section is to look at the basic concepts associated with all functions.



Prerequisites

Before starting this Section you should ...

- have a thorough understanding of basic algebraic notation and techniques



Learning Outcomes

On completion you should be able to ...

- explain what is meant by a function
- use common notations for functions
- explain what is meant by the argument of a function

1. The function rule

A function can be thought of as a **rule** which operates on an **input** and produces an **output**. This is often illustrated pictorially in two ways as shown in Figure 1. The first way is by using a **block diagram** which consists of a box showing the input, the output and the rule. We often write the rule inside the box. The second way is to use two sets, one to represent the input and one to represent the output with an arrow showing the relationship between them.

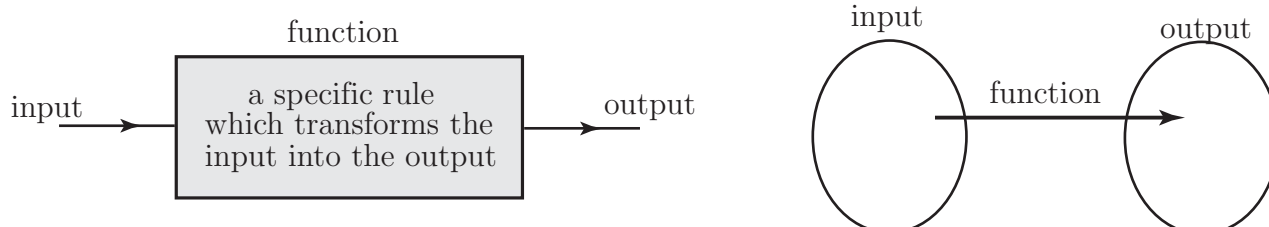


Figure 1: A general function

More precisely, a rule is a function if it produces only a **single** output for any given input. The function with the rule 'treble the input' is shown in Figure 2.

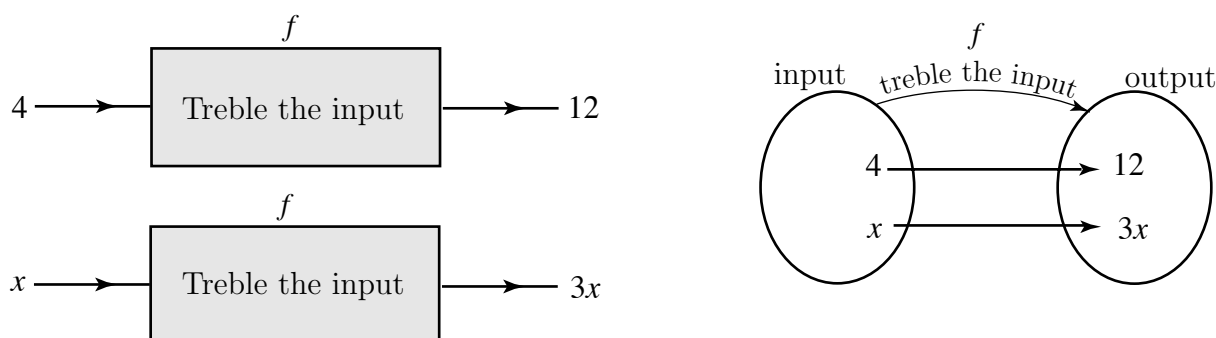


Figure 2: The function with the rule 'treble the input'

Note that with an input of 4 the function will produce an output of 12. With a more general input, x say, the output will be $3x$. It is usual to assign a letter or other symbol to a function in order to label it. The trebling function in Figure 2 has been given the symbol f .



Key Point 1

A function is a rule which operates on an input and produces a **single** output from that input.



Write down the output from the function shown in Figure 3 when the input is
(a) 4, (b) -3 , (c) x (d) t .

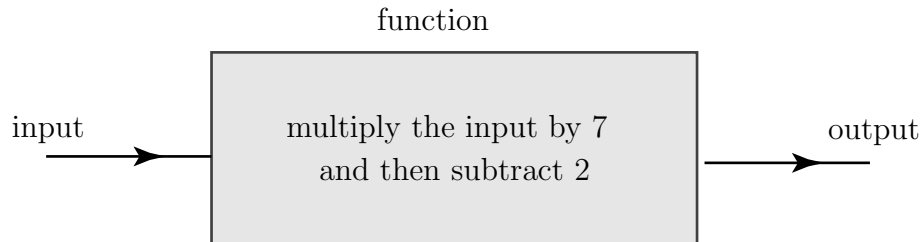


Figure 3

Your solution

In each case the function rule instructs you to multiply the input by 7 and then subtract 2. Evaluate the corresponding outputs.

Answer

- (a) When the input is 4 the output is 26
- (b) When the input is -3 the output is -23
- (c) When the input is x the output is $7x - 2$
- (d) When the input is t the output is $7t - 2$.

Several different notations are used by engineers to describe functions. For the trebling function in Figure 2 it is common to write

$$f(x) = 3x$$

This indicates that with an input x , the function, f , produces an output of $3x$. The input to the function is placed in the brackets after the ' f '. $f(x)$ is read as ' f is a function of x ', or simply ' f of x ', meaning that the value of the output from the function depends upon the value of the input x . The value of the output is often called the 'value of the function'.

**Example 1**

State in words the rule defined by each of the following functions:

(a) $f(x) = 6x$

(b) $f(t) = 6t - 1$

(c) $g(x) = x^2 - 7$

(d) $h(t) = t^3 + 5$

(e) $p(x) = x^3 + 5$

Solution

(a) The rule for f is 'multiply the input by 6'.

(b) Here the input has been labelled t . The rule for f is 'multiply the input by 6 and subtract 1'.

(c) Here the function has been labelled g . The rule for g is 'square the input and subtract 7'.

(d) The rule for h is 'cube the input and add 5'.

(e) The rule for p is 'cube the input and add 5'.

Note from Example 1, parts (d) and (e), that it is the rule that is important when describing a function and not the letters used. Both $h(t)$ and $p(x)$ instruct us to 'cube the input and add 5'.

**Task**

Write down a mathematical function which can be used to describe the following rules:

(a) 'square the input and divide the result by 2'. Use the letter x for input and the letter f to represent the function.

(b) 'divide the input by 3 and add 7'. Call the function g and call the input t .

Your solution**Answer**

$$(a) f(x) = \frac{x^2}{2}, \quad (b) g(t) = \frac{t}{3} + 7$$

Exercise

State the rule of each of the following functions:

(a) $f(x) = 5x$, (b) $f(t) = 5t$, (c) $f(x) = 8x + 10$, (d) $f(t) = 7t - 27$, (e) $f(t) = 1 - t$,

(f) $h(t) = \frac{t}{3} + \frac{2}{3}$, (g) $f(x) = \frac{1}{1+x}$

Answers

(a) multiply the input by 5. (b) same as (a). (c) multiply the input by 8 and then add 10. (d) multiply the input by 7 and then subtract 27. (e) subtract the input from 1. (f) divide the input by 3 and then add $\frac{2}{3}$. (g) add 1 to the input and then find the reciprocal of the result.

2. The argument of a function

The input to a function is sometimes called its **argument**. It is frequently necessary to obtain the output from a function if we are given its argument. For example, given the function $g(t) = 3t + 2$ we may require the value of the output when the argument is 4. We write this as $g(t = 4)$ or more usually and compactly as $g(4)$. In this case the value of $g(4)$ is $3 \times 4 + 2 = 14$.



Example 2

Given the function $f(x) = 3x + 1$ find

- (a) $f(2)$
- (b) $f(-1)$
- (c) $f(6)$

Solution

- (a) The output from the function needs to be found when the argument or input is 2. We need to replace x by 2 in the expression for the function. We find

$$f(2) = 3 \times 2 + 1 = 7$$

- (b) Here the argument is -1 . We find

$$f(-1) = 3 \times (-1) + 1 = -2$$

- (c) $f(6) = 3 \times 6 + 1 = 19$.



Given the function $g(t) = 6t + 4$ find (a) $g(3)$, (b) $g(6)$, (c) $g(-2)$

Your solution

Answer

a) $g(3) = 6 \times 3 + 4 = 22$, (b) $g(6) = 40$, (c) $g(-2) = -8$

It is possible to obtain the value of a function when the argument is an algebraic expression. Consider the following Example.



Example 3

Given the function $y(x) = 3x + 2$ find

- (a) $y(t)$
- (b) $y(2t)$
- (c) $y(z + 2)$
- (d) $y(5x)$

Solution

The rule for this function is 'multiply the input by 3 and then add 2'. We can apply this rule whatever the argument.

- (a) In this case the argument is t . Multiplying this by 3 and adding 2 we find $y(t) = 3t + 2$. Equivalently we can replace x by t in the expression for the function, so, $y(t) = 3t + 2$.
- (b) In this case the argument is $2t$. We need to replace x by $2t$ in the expression for the function. So $y(2t) = 3(2t) + 2 = 6t + 2$
- (c) In this case the argument is $z + 2$. We find $y(z + 2) = 3(z + 2) + 2 = 3z + 8$. It is important to note that $y(z + 2)$ is **not** $y \times (z + 2) = yz + y2$ but instead reads 'y of $(z + 2)$ ' where 'of' means 'take the function of'.
- (d) Here we have a complication. The argument is $5x$ and so there appears to be a clash of notation with the original expression for the function. There is no problem if we remember that the rule is to multiply the input by 3 and then add 2. The input now is $5x$. So $y(5x) = 3(5x) + 2 = 15x + 2$.



Given the function $g(x) = 8 - 2x$ find (a) $g(4)$, (b) $g(4t)$, (c) $g(2x - 3)$

Your solution

- (a)
- (b)
- (c)

Answer

(a) $g(4) = 8 - 2 \times 4 = 0$

(b) $g(4t) = 8 - 2 \times 4t = 8 - 8t$

(c) $g(2x - 3) = 8 - 2(2x - 3) = 14 - 4x$

Exercises

1. Explain what is meant by the 'argument' of a function.
2. Given the function $g(t) = 8t + 3$ find (a) $g(7)$, (b) $g(2)$, (c) $g(-0.5)$, (d) $g(-0.11)$
3. Given the function $f(t) = 2t^2 + 4$ find: (a) $f(x)$ (b) $f(2x)$ (c) $f(-x)$ (d) $f(4x + 2)$
(e) $f(3t + 5)$ (f) $f(\lambda)$ (g) $f(t - \lambda)$ (h) $f(\frac{t}{\alpha})$
4. Given $g(x) = 3x^2 - 7$ find (a) $g(3t)$, (b) $g(t + 5)$, (c) $g(6t - 4)$, (d) $g(4x + 9)$
5. Calculate $f(x + h)$ when (a) $f(x) = x^2$, (b) $f(x) = x^3$, (c) $f(x) = \frac{1}{x}$. In each case write down the corresponding expression for $f(x + h) - f(x)$.
6. If $f(x) = \frac{1}{(1 - x)^2}$ find $f(\frac{x}{\ell})$.

Answers

1. The argument is the input.
2. (a) 59, (b) 19, (c) -1, (d) 2.12.
3. (a) $2x^2 + 4$, (b) $8x^2 + 4$, (c) $2x^2 + 4$, (d) $32x^2 + 32x + 12$, (e) $18t^2 + 60t + 54$,
(f) $2\lambda^2 + 4$, (g) $2(t - \lambda)^2 + 4$, (h) $\frac{2t^2}{\alpha^2} + 4$.
4. (a) $27t^2 - 7$, (b) $3t^2 + 30t + 68$, (c) $108t^2 - 144t + 41$, (d) $48x^2 + 216x + 236$.
5. (a) $x^2 + 2xh + h^2$, (b) $x^3 + 3x^2h + 3xh^2 + h^3$, (c) $\frac{1}{x + h}$.
The corresponding expressions are (a) $2xh + h^2$, (b) $3x^2h + 3xh^2 + h^3$,
(c) $\frac{1}{x + h} - \frac{1}{x} = -\frac{h}{x(x + h)}$.
6. $\frac{1}{(1 - \frac{x}{\ell})^2}$.

3. Composition of functions

Consider the two functions $g(x) = x^2$, and $h(x) = 3x + 5$. Block diagrams showing the rules for these functions are shown in Figure 4.

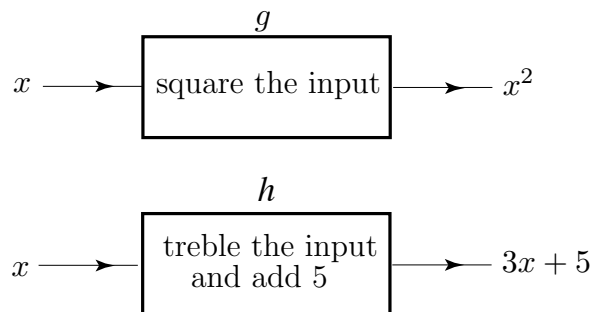


Figure 4: Block diagrams of two functions g and h

Suppose we place these Block diagrams together in series as shown in Figure 5, so that the output from function g is used as the input to function h .

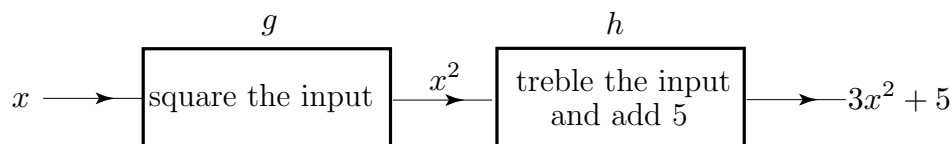


Figure 5: The composition of the two functions to give $h(g(x))$

Study Figure 5 carefully and deduce that when the input to g is x the output from the two functions in series is $3x^2 + 5$. Since the output from g is used as input to h we write

$$h(g(x)) = h(x^2) = 3x^2 + 5$$

The form $h(g(x))$ is known as the **composition** of the functions g and h .

Suppose we interchange the two functions so that h is applied first as shown in Figure 6.

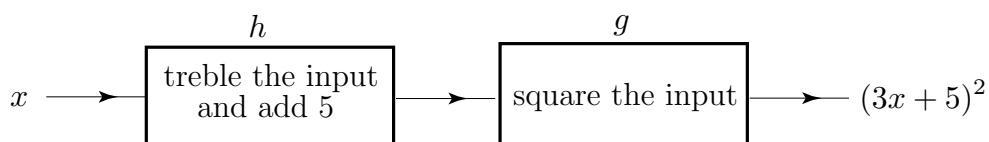


Figure 6: The composition of the two functions to give $g(h(x))$

Study Figure 6 and note that when the input to h is x the final output is $(3x + 5)^2$. We write

$$g(h(x)) = (3x + 5)^2$$

Note that the function $h(g(x))$ is different from $g(h(x))$.



Example 4

Given two functions $g(t) = 3t + 2$ and $h(t) = t + 3$ obtain an expression for the composition $g(h(t))$.

Solution

We have $g(h(t)) = g(t + 3)$. Now the rule for g is 'triple the input and add 2', and so we can write $g(t + 3) = 3(t + 3) + 2 = 3t + 11$ so, $g(h(t)) = 3t + 11$.



Given the two functions $g(t) = 3t + 2$ and $h(t) = t + 3$ as in Example 4 above, obtain an expression for the composition $h(g(t))$.

Your solution

We have

$$h(g(t)) = h(3t + 2)$$

State the rule for h and write down $h(g(t))$.

Answer

'add 3 to the input', $h(3t + 2) = 3t + 5$. Note that $h(g(t)) \neq g(h(t))$.

Exercises

1. Find $f(g(x))$ when $f(x) = x - 7$ and $g(x) = x^2$.
2. If $f(x) = 8x + 2$ find $f(f(x))$.
3. If $f(x) = x + 6$ and $g(x) = x^2 - 5$ find (a) $f(g(0))$, (b) $g(f(0))$, (c) $g(g(2))$, (d) $f(g(7))$.
4. If $f(x) = \frac{x-3}{x+1}$ and $g(x) = \frac{1}{x}$ find $g(f(x))$.

Answers

1. $x^2 - 7$.
2. $8(8x + 2) + 2 = 64x + 18$.
3. (a) 1, (b) 31, (c) -4, (d) 50.
4. $\frac{x+1}{x-3}$.