Contents 1

Basic Algebra

Learning outcomes

In this Workbook you will learn about some of the basic building blocks of mathematics. As well as becoming familiar with the notation and symbols used in mathematics you will learn the fundamental rules of algebra upon which much of mathematics is based. In particular you will learn about indices and how to simplify algebraic expressions, using a variety of approaches: collecting like terms, removing brackets and factorisation. Finally, you will learn how to transpose formulae.

Mathematical Notation and Symbols

1.1

Introduction

This introductory Section reminds you of important notations and conventions used throughout engineering mathematics. We discuss the arithmetic of numbers, the plus or minus sign, $\pm$, the modulus notation $|\ |$, and the factorial notation !. We examine the order in which arithmetical operations are carried out. Symbols are introduced to represent physical quantities in formulae and equations. The topic of algebra deals with the manipulation of these symbols. The Section closes with an introduction to algebraic conventions. In what follows a working knowledge of the addition, subtraction, multiplication and division of numerical fractions is essential.

Prerequisites

Before starting this Section you should ...

- be able to add, subtract, multiply and divide fractions
- be able to express fractions in equivalent forms

Learning Outcomes

On completion you should be able to ...

- recognise and use a wide range of common mathematical symbols and notations

1. Numbers, operations and common notations

A knowledge of the properties of numbers is fundamental to the study of engineering mathematics. Students who possess this knowledge will be well-prepared for the study of algebra. Much of the terminology used throughout the rest of this Section can be most easily illustrated by applying it to numbers. For this reason we strongly recommend that you work through this Section even if the material is familiar.

The number line

A useful way of picturing numbers is to use a **number line**. Figure 1 shows part of this line. Positive numbers are represented on the right-hand side of this line, negative numbers on the left-hand side. Any whole or fractional number can be represented by a point on this line which is also called the **real number line**, or simply the **real line**. Study Figure 1 and note that a minus sign is always used to indicate that a number is negative, whereas the use of a plus sign is optional when describing positive numbers.

The line extends indefinitely both to the left and to the right. Mathematically we say that the line extends from minus infinity to plus infinity. The symbol for **infinity** is ∞.

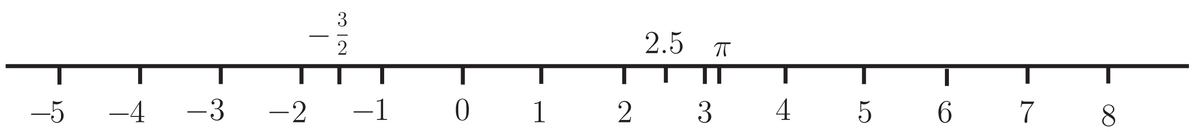


Figure 1: Numbers can be represented on a number line

The symbol $>$ means 'greater than'; for example $6 > 4$. Given any number, all numbers to the right of it on the number line are greater than the given number. The symbol $<$ means 'less than'; for example $-3 < 19$. We also use the symbols $\geq$ meaning 'greater than or equal to' and $\leq$ meaning 'less than or equal to'. For example, $7 \leq 10$ and $7 \leq 7$ are both true statements.

Sometimes we are interested in only a small section, or **interval**, of the real line. We write $[1, 3]$ to denote all the real numbers between 1 and 3 inclusive, that is 1 and 3 are included in the interval. Therefore the interval $[1, 3]$ consists of all real numbers x, such that $1 \leq x \leq 3$. The square brackets, $[,]$ mean that the end-points are included in the interval and such an interval is said to be **closed**. We write $(1, 3)$ to represent all real numbers between 1 and 3, but not including the end-points. Thus $(1, 3)$ means all real numbers x such that $1 < x < 3$, and such an interval is said to be **open**. An interval may be closed at one end and open at the other. For example, $(1, 3]$ consists of all numbers x such that $1 < x \leq 3$. Intervals can be represented on a number line. A **closed end-point** is denoted by $\bullet$; an **open end-point** is denoted by $\circ$. The intervals $(-6, -4)$, $[-1, 2]$ and $(3, 4]$ are illustrated in Figure 2.

Figure 2: The intervals $(-6, -4)$, $[-1, 2]$ and $(3, 4]$ depicted on the real line

2. Calculation with numbers

To perform calculations with numbers we use the **operations**, $+$, $-$, $\times$ and $\div$.

Addition ($+$)

We say that $4+5$ is the **sum** of 4 and 5. Note that $4+5$ is equal to $5+4$ so that the order in which we write down the numbers does not matter when we are adding them. Because the order does not matter, addition is said to be **commutative**. This first property is called **commutativity**.

When more than two numbers are to be added, as in $4+8+9$, it makes no difference whether we add the 4 and 8 first to get $12+9$, or whether we add the 8 and 9 first to get $4+17$. Whichever way we work we will obtain the same result, 21. Addition is said to be **associative**. This second property is called **associativity**.

Subtraction ($-$)

We say that $8-3$ is the **difference** of 8 and 3. Note that $8-3$ is *not* the same as $3-8$ and so the order in which we write down the numbers is important when we are subtracting them i.e. subtraction is not commutative. Subtracting a negative number is equivalent to adding a positive number, thus $7-(-3) = 7+3 = 10$.

The plus or minus sign ($\pm$)

In engineering calculations we often use the notation **plus or minus**, $\pm$. For example, we write 12 ± 8 as shorthand for the two numbers $12+8$ and $12-8$, that is 20 and 4. If we say a number lies in the range 12 ± 8 we mean that the number can lie between 4 and 20 inclusive.

Multiplication ($\times$)

The instruction to multiply, or obtain the product of, the numbers 6 and 7 is written 6×7. Sometimes the multiplication sign is missed out altogether and we write $(6)(7)$.

Note that $(6)(7)$ is the same as $(7)(6)$ so multiplication of numbers is commutative. If we are multiplying three numbers, as in $2\times 3\times 4$, we obtain the same result whether we multiply the 2 and 3 first to obtain 6×4, or whether we multiply the 3 and 4 first to obtain 2×12. Either way the result is 24. Multiplication of numbers is associative.

Recall that when multiplying positive and negative numbers the sign of the result is given by the rules given in Key Point 1.

Key Point 1

Multiplication

When multiplying numbers:

positive $\times$ positive $=$ positive   negative $\times$ negative $=$ positive
positive $\times$ negative $=$ negative   negative $\times$ positive $=$ negative

For example, $(-4) \times 5 = -20$, and $(-3) \times (-6) = 18$.

When dealing with fractions we sometimes use the word 'of' as in 'find $\frac{1}{2}$ of 36'. In this context 'of' is equivalent to multiply, that is

$$\frac{1}{2} \text{ of } 36 \quad \text{is equivalent to} \quad \frac{1}{2} \times 36 = 18$$

Division (÷) or (/)

The quantity $8 \div 4$ means 8 divided by 4. This is also written as $8/4$ or $\frac{8}{4}$ and is known as the **quotient** of 8 and 4. In the fraction $\frac{8}{4}$ the top line is called the **numerator** and the bottom line is called the **denominator**. Note that $8/4$ is not the same as $4/8$ and so the order in which we write down the numbers is important. Division is not commutative.

When dividing positive and negative numbers, recall the following rules in Key Point 2 for determining the sign of the result:

Key Point 2

Division

When dividing numbers:

$$\frac{\text{positive}}{\text{positive}} = \text{positive} \qquad \frac{\text{positive}}{\text{negative}} = \text{negative}$$

$$\frac{\text{negative}}{\text{positive}} = \text{negative} \qquad \frac{\text{negative}}{\text{negative}} = \text{positive}$$

The reciprocal of a number

The **reciprocal** of a number is found by inverting it. If the number $\frac{2}{3}$ is **inverted** we get $\frac{3}{2}$. So the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. Because we can write 4 as $\frac{4}{1}$, the reciprocal of 4 is $\frac{1}{4}$.

State the reciprocal of (a) $\frac{6}{11}$, (b) $\frac{1}{5}$, (c) -7.

Your solution

(a) (b) (c)

Answer

The modulus notation (| |)

We shall make frequent use of the modulus notation $|\ |$. The **modulus** of a number is the size of that number regardless of its sign. For example $|4|$ is equal to 4, and $|-3|$ is equal to 3. The modulus of a number is thus never negative.

State the modulus of (a) -17, (b) $\frac{1}{5}$, (c) $-\frac{1}{7}$ (d) 0.

Your solution

(a) (b) (c) (d)

Answer

The modulus of a number is found by ignoring its sign. (a) 17 (b) $\frac{1}{5}$ (c) $\frac{1}{7}$ (d) 0

The factorial symbol (!)

Another commonly used notation is the **factorial**, denoted by the exclamation mark '!'. The number $5!$, read 'five factorial', or 'factorial five', is a shorthand notation for the expression $5 \times 4 \times 3 \times 2 \times 1$, and the number $7!$ is shorthand for $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$. Note that $1!$ equals 1, and by convention $0!$ is defined as 1 also. Your scientific calculator is probably able to evaluate factorials of small integers. It is important to note that factorials only apply to positive integers.

Key Point 3

Factorial notation

If n is a positive integer then $n! = n \times (n-1) \times (n-2) \ldots 5 \times 4 \times 3 \times 2 \times 1$

Example 1

(a) Evaluate 4! and 5! without using a calculator.

(b) Use your calculator to find $10!$.

Solution

(a) $4! = 4 \times 3 \times 2 \times 1 = 24$. Similarly, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. Note that $5! = 5 \times 4! = 5 \times 24 = 120$.

(b) $10! = 3,628,800$.

Task

Find the factorial button on your calculator and hence compute 11!.
(The button may be marked ! or $n!$). Check that $11! = 11 \times 10!$

Your solution

$11! =$ $\qquad$ $11 \times 10! =$

Answer

$11! = 39916800$

$11 \times 10! = 11 \times 3628800 = 39916800$

3. Rounding to n decimal places

In general, a calculator or computer is unable to store every decimal place of a real number. Real numbers are **rounded**. To round a number to n decimal places we look at the $(n+1)^{th}$ digit in the decimal expansion of the number.

- If the $(n+1)^{th}$ digit is 0, 1, 2, 3 or 4 then we **round down**: that is, we simply chop to n places. (In other words we neglect the $(n+1)^{th}$ digit and any digits to its right.)
- If the $(n+1)^{th}$ digit is 5, 6, 7, 8 or 9 then we **round up**: we add 1 to the n^{th} decimal place and then chop to n places.

For example

$$\frac{1}{3} = 0.3333 \qquad \text{rounded to 4 decimal places}$$

$$\frac{8}{3} = 2.66667 \qquad \text{rounded to 5 decimal places}$$

$$\pi = 3.142 \qquad \text{rounded to 3 decimal places}$$

$$2.3403 = 2.340 \qquad \text{rounded to 3 decimal places}$$

Sometimes the phrase 'decimal places' is abbreviated to 'd.p.' or 'dec.pl.'.

Example 2

Write down each of these numbers rounded to 4 decimal places:
$0.12345,\ -0.44444,\ 0.5555555,\ 0.000127351,\ 0.000005,\ 123.456789$

Solution

$0.1235,\ -0.4444,\ 0.5556,\ 0.0001,\ 0.0000,\ 123.4568$

Task

Write down each of these numbers, rounded to 3 decimal places:
$0.87264,\ 0.1543,\ 0.889412,\ -0.5555,\ 45.6789,\ 6.0003$

Your solution

Answer

$0.873,\ 0.154,\ 0.889,\ -0.556,\ 45.679,\ 6.000$

4. Rounding to n significant figures

This process is similar to rounding to decimal places but there are some subtle differences.

To round a number to n significant figures we look at the $(n+1)^{th}$ digit in the decimal expansion of the number.

- If the $(n+1)^{th}$ digit is 0, 1, 2, 3 or 4 then we **round down**: that is, we simply chop to n places, inserting zeros if necessary before the decimal point. (In other words we neglect the $(n+1)^{th}$ digit and any digits to its right.)

- If the $(n+1)^{th}$ digit is 5, 6, 7, 8 or 9 then we **round up**: we add 1 to the n^{th} decimal place and then chop to n places, inserting zeros if necessary before the decimal point.

Examples are given on the next page.

$$\frac{1}{3} = 0.3333 \quad \text{rounded to 4 significant figures}$$

$$\frac{8}{3} = 2.66667 \quad \text{rounded to 6 significant figures}$$

$$\pi = 3.142 \quad \text{rounded to 4 significant figures}$$

$$2136 = 2000 \quad \text{rounded to 1 significant figure}$$

$$36.78 = 37 \quad \text{rounded to 2 significant figures}$$

$$6.2399 = 6.240 \quad \text{rounded to 4 significant figures}$$

Sometimes the phrase "significant figures" is abbreviated as "s.f." or "sig.fig."

Example 3

Write down each of these numbers, rounding them to 4 significant figures:
0.12345, -0.44444, 0.5555555, 0.000127351, 25679, 123.456789, 3456543

Solution

0.1235, -0.4444, 0.5556, 0.0001274, 25680, 123.5, 3457000

Task

Write down each of these numbers rounded to 3 significant figures:
0.87264, 0.1543, 0.889412, -0.5555, 2.346, 12343.21, 4245321

Your solution

Answer

0.873, 0.154, 0.889, -0.556, 2.35, 12300, 4250000

Arithmetical expressions

A quantity made up of numbers and one or more of the operations $+$, $-$, $\times$ and $/$ is called an **arithmetical expression**. Frequent use is also made of brackets, or **parentheses**, ( ), to separate different parts of an expression. When evaluating an expression it is conventional to evaluate quantities within brackets first. Often a division line implies bracketed quantities. For example in the expression $\dfrac{3+4}{7+9}$ there is implied bracketing of the numerator and denominator i.e. the expression is $\dfrac{(3+4)}{(7+9)}$ and the bracketed quantities would be evaluated first resulting in the number $\dfrac{7}{16}$.

The BODMAS rule

When several arithmetical operations are combined in one expression we need to know in which order to perform the calculation. This order is found by applying rules known as **precedence rules** which specify which operation has priority. The convention is that bracketed expressions are evaluated first. Any multiplications and divisions are then performed, and finally any additions and subtractions. For short, this is called the BODMAS rule.

Key Point 4

The BODMAS rule

Brackets, ( )	First priority: evaluate terms within brackets
Of, $\times$ **D**ivision, $\div$ **M**ultiplication, $\times$	Second priority: carry out all multiplications and divisions
Addition, $+$ **S**ubtraction, $-$	Third priority: carry out all additions and subtractions

If an expression contains only multiplication and division we evaluate by working from left to right. Similarly, if an expression contains only addition and subtraction we evaluate by working from left to right. In Section 1.2 we will meet another operation called exponentiation, or raising to a power. We shall see that, in the simplest case, this operation is repeated multiplication and it is usually carried out once any brackets have been evaluated.

Example 4

Evaluate $4 - 3 + 7 \times 2$

Solution

The BODMAS rule tells us to perform the multiplication before the addition and subtraction. Thus

$$4 - 3 + 7 \times 2 = 4 - 3 + 14$$

Finally, because the resulting expression contains just addition and subtraction we work from the left to the right, that is

$$4 - 3 + 14 = 1 + 14 = 15$$

Task

Evaluate $4+3\times 7$ using the BODMAS rule to decide which operation to carry out first.

Your solution

$4+3\times 7=$

Answer

25 (Multiplication has a higher priority than addition.)

Task

Evaluate $(4-2)\times 5$.

Your solution

$(4-2)\times 5=$

Answer

$2\times 5=10$. (The bracketed quantity must be evaluated first.)

Example 5

Evaluate $8\div 2-(4-5)$

Solution

The bracketed expression is evaluated first:

$$8\div 2-(4-5)=8\div 2-(-1)$$

Division has higher priority than subtraction and so this is carried out next giving

$$8\div 2-(-1)=4-(-1)$$

Subtracting a negative number is equivalent to adding a positive number. Thus

$$4-(-1)=4+1=5$$

Task

Evaluate $\dfrac{9-4}{25-5}$.
(Remember that the dividing line implies that brackets are present around the numerator and around the denominator.)

Your solution

Answer
$\dfrac{9-4}{25-5} = \dfrac{(9-4)}{(25-5)} = \dfrac{5}{20} = \dfrac{1}{4}$

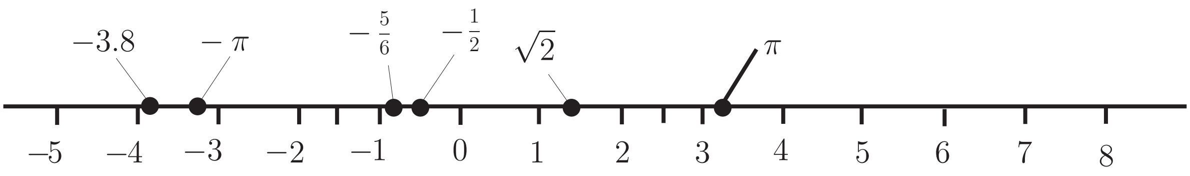
Exercises

1. Draw a number line and on it label points to represent -5, -3.8, $-\pi$, $-\dfrac{5}{6}$, $-\dfrac{1}{2}$, 0, $\sqrt{2}$, π, 5.
2. Simplify without using a calculator (a) -5×-3, (b) -5×3, (c) 5×-3, (d) 15×-4, (e) -14×-3, (f) $\dfrac{18}{-3}$, (g) $\dfrac{-21}{7}$, (h) $\dfrac{-36}{-12}$.
3. Evaluate (a) $3 + 2 \times 6$, (b) $3 - 2 - 6$, (c) $3 + 2 - 6$, (d) $15 - 3 \times 2$, (e) $15 \times 3 - 2$, (f) $(15 \div 3) + 2$, (g) $15 \div 3 + 2$, (h) $7 + 4 - 11 - 2$, (i) $7 \times 4 + 11 \times 2$, (j) $-(-9)$, (k) $7 - (-9)$, (l) $-19 - (-7)$, (m) $-19 + (-7)$.
4. Evaluate (a) $|-18|$, (b) $|4|$, (c) $|-0.001|$, (d) $|0.25|$, (e) $|0.01 - 0.001|$, (f) $2!$, (g) $8! - 3!$, (h) $\dfrac{9!}{8!}$.
5. Evaluate (a) $8 + (-9)$, (b) $18 - (-8)$, (c) $-18 + (-2)$, (d) $-11 - (-3)$
6. State the reciprocal of (a) 8, (b) $\dfrac{9}{13}$.
7. Evaluate (a) 7 ± 3, (b) 16 ± 7, (c) $-15 \pm \dfrac{1}{2}$, (d) -16 ± 0.05, (e) $|-8| \pm 13$, (f) $|-2| \pm 8$.
8. Which of the following statements are true ?

   (a) $-8 \leq 8$, (b) $-8 \leq -8$, (c) $-8 \leq |8|$, (d) $|-8| < 8$, (e) $|-8| \leq -8$, (f) $9! \leq 8!$, (g) $8! \leq 10!$.
9. Explain what is meant by saying that addition of numbers is (a) associative, (b) commutative. Give examples.
10. Explain what is meant by saying that multiplication of numbers is (a) associative, (b) commutative. Give examples.

Answers

1.



2. (a) 15, (b) -15, (c) -15, (d) -60, (e) 42, (f) -6, (g) -3, (h) 3.
3. (a) 15, (b) -5, (c) -1, (d) 9, (e) 43, (f) 7, (g) 7, (h) -2, (i) 50, (j) 9, (k) 16, (l) -12, (m) -26
4. (a) 18, (b) 4, (c) 0.001, (d) 0.25, (e) 0.009, (f) 2, (g) 40314, (h) 9,
5. (a) -1, (b) 26, (c) -20, (d) -8
6. (a) $\frac{1}{8}$, (b) $\frac{13}{9}$.
7. (a) 4,10, (b) 9,23, (c) $-15\frac{1}{2}$, $-14\frac{1}{2}$, (d) $-16.05, -15.95$, (e) $-5, 21$, (f) $-6, 10$
8. (a), (b), (c), (g) are true.
9. For example (a) $(1+2)+3 = 1+(2+3)$, and both are equal to 6. (b) $8+2 = 2+8$.
10. For example (a) $(2\times 6)\times 8 = 2\times(6\times 8)$, and both are equal to 96. (b) $7\times 5 = 5\times 7$.

5. Using symbols

Mathematics provides a very rich language for the communication of engineering concepts and ideas, and a set of powerful tools for the solution of engineering problems. In order to use this language it is essential to appreciate how **symbols** are used to represent physical quantities, and to understand the rules and conventions which have been developed to manipulate these symbols.

The choice of which letters or other symbols to use is largely up to the user although it is helpful to choose letters which have some meaning in any particular context. For instance if we wish to choose a symbol to represent the temperature in a room we might use the capital letter T. Similarly the lower case letter t is often used to represent time. Because both time and temperature can vary we refer to T and t as **variables**.

In a particular calculation some symbols represent fixed and unchanging quantities and we call these **constants**. Often we reserve the letters x, y and z to stand for variables and use the earlier letters of the alphabet, such as a, b and c, to represent constants. The Greek letter pi, written π, is used to represent the constant 3.14159.... which appears for example in the formula for the area of a circle. Other Greek letters are frequently used as symbols, and for reference, the Greek alphabet is given in Table 1.

Table 1: The Greek alphabet

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	Λ	λ	lambda	T	τ	tau
Γ	γ	gamma	K	κ	kappa	Σ	σ	sigma
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Mathematics is a very precise language and care must be taken to note the exact position of any symbol in relation to any other. If x and y are two symbols, then the quantities xy, x^y, x_y can all mean different things. In the expression x^y you will note that the symbol y is placed to the right of and slightly higher than the symbol x. In this context y is called a **superscript**. In the expression x_y, y is placed lower than and to the right of x, and is called a **subscript**.

Example The temperature in a room is measured at four points as shown in Figure 3.

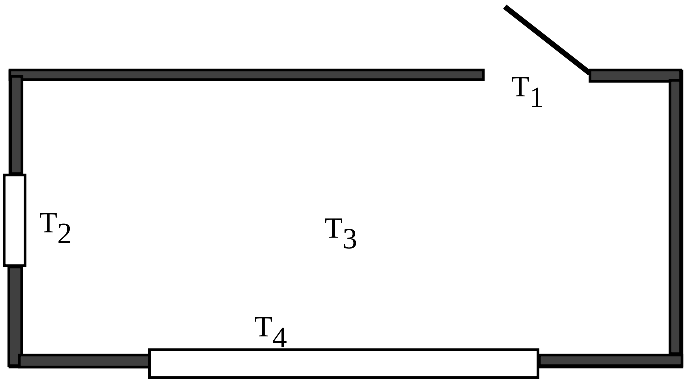


Figure 3: The temperature is measured at four points

Rather than use different letters to represent the four measurements we can use one symbol, T, together with four subscripts to represent the temperature. Thus the four measurements are denoted by T_1, T_2, T_3 and T_4.

6. Combining numbers together using $+, -, \times, \div$

Addition ($+$)

If the letters x and y represent two numbers, then their **sum** is written as $x + y$. Note that $x + y$ is the same as $y + x$ just as $4 + 7$ is equal to $7 + 4$.

Subtraction ($-$)

Subtracting y from x yields $x - y$. Note that $x - y$ is not the same as $y - x$ just as $11 - 7$ is not the same as $7 - 11$, however in both cases the difference is said to be 4.

Multiplication ($\times$)

The instruction to multiply x and y together is written as $x \times y$. Usually the multiplication sign is omitted and we write simply xy. An alternative notation is to use a dot to represent multiplication and so we could write $x.y$ The quantity xy is called the **product** of x and y. As discussed earlier multiplication is both commutative and associative:

i.e. $\quad x \times y = y \times x \quad$ and $\quad (x \times y) \times z = x \times (y \times z)$

This last expression can thus be written $x \times y \times z$ without ambiguity. When mixing numbers and symbols it is usual to write the numbers first. Thus $3 \times x \times y \times 4 = 3 \times 4 \times x \times y = 12xy$.

Example 6

Simplify (a) $9(2y)$, (b) $-3(5z)$, (c) $4(2a)$, (d) $2x \times (2y)$.

Solution

(a) Note that $9(2y)$ means $9\times(2\times y)$. Because of the associativity of multiplication $9\times(2\times y)$ means the same as $(9 \times 2) \times y$, that is $18y$.

(b) $-3(5z)$ means $-3 \times (5 \times z)$. Because of associativity this is the same as $(-3 \times 5) \times z$, that is $-15z$.

(c) $4(2a)$ means $4 \times (2 \times a)$. We can write this as $(4 \times 2) \times a$, that is $8a$.

(d) Because of the associativity of multiplication, the brackets are not needed and we can write $2x \times (2y) = 2x \times 2y$ which equals

$$2 \times x \times 2 \times y = 2 \times 2 \times x \times y = 4xy.$$

Example 7

What is the distinction between $9(-2y)$ and $9 - 2y$?

Solution

The expression $9(-2y)$ means $9 \times (-2y)$. Because of associativity of multiplication we can write this as $9 \times (-2) \times y$ which equals $-18y$.

On the other hand $9 - 2y$ means subtract $2y$ from 9. This cannot be simplified.

Division (÷)

The quantity $x \div y$ means x divided by y. This is also written as x/y or $\frac{x}{y}$ and is known as the **quotient** of x and y. In the expression $\frac{x}{y}$ the symbol x is called the **numerator** and the symbol y is called the **denominator**. Note that x/y is not the same as y/x. Division by 1 leaves a quantity unchanged so that $\frac{x}{1}$ is simply x.

Algebraic expressions

A quantity made up of symbols and the operations $+$, $-$, $\times$ and / is called an **algebraic expression**. One algebraic expression divided by another is called an algebraic fraction. Thus

$$\frac{x+7}{x-3} \qquad \text{and} \qquad \frac{3x-y}{2x+z}$$

are algebraic fractions. The **reciprocal** of an algebraic fraction is found by inverting it. Thus the reciprocal of $\frac{2}{x}$ is $\frac{x}{2}$. The reciprocal of $\frac{x+7}{x-3}$ is $\frac{x-3}{x+7}$.

Example 8

State the reciprocal of each of the following expressions:

(a) $\frac{y}{z}$, (b) $\frac{x+z}{a-b}$, (c) $3y$, (d) $\frac{1}{a+2b}$, (e) $-\frac{1}{y}$

Solution

(a) $\frac{z}{y}$.

(b) $\frac{a-b}{x+z}$.

(c) $3y$ is the same as $\frac{3y}{1}$ so the reciprocal of $3y$ is $\frac{1}{3y}$.

(d) The reciprocal of $\frac{1}{a+2b}$ is $\frac{a+2b}{1}$ or simply $a+2b$.

(e) The reciprocal of $-\frac{1}{y}$ is $-\frac{y}{1}$ or simply $-y$.

Finding the reciprocal of complicated expressions can cause confusion. Study the following Example carefully.

Example 9

Obtain the reciprocal of:

(a) $p + q$, (b) $\dfrac{1}{R_1} + \dfrac{1}{R_2}$

Solution

(a) Because $p + q$ can be thought of as $\dfrac{p+q}{1}$ its reciprocal is $\dfrac{1}{p+q}$. Note in particular that the reciprocal of $p + q$ is **not** $\dfrac{1}{p} + \dfrac{1}{q}$. This distinction is important and a common cause of error. To avoid an error carefully identify the numerator and denominator in the original expression before inverting.

(b) The reciprocal of $\dfrac{1}{R_1} + \dfrac{1}{R_2}$ is $\dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$. To simplify this further requires knowledge of the addition of algebraic fractions which is dealt with in HELM 1.4. It is important to note that the reciprocal of $\dfrac{1}{R_1} + \dfrac{1}{R_2}$ is **not** $R_1 + R_2$.

The equals sign (=)

The equals sign, $=$, is used in several different ways.

Firstly, an equals sign is used in **equations**. The left-hand side and right-hand side of an equation are equal only when the variable involved takes specific values known as **solutions** of the equation. For example, in the equation $x - 8 = 0$, the variable is x. The left-hand side and right-hand side are only equal when x has the value 8. If x has any other value the two sides are not equal.

Secondly, the equals sign is used in **formulae**. Physical quantities are often related through a formula. For example, the formula for the length, C, of the circumference of a circle expresses the relationship between the circumference of the circle and its radius, r. This formula states $C = 2\pi r$. When used in this way the equals sign expresses the fact that the quantity on the left is found by evaluating the expression on the right.

Thirdly, an equals sign is used in **identities**. An identity looks just like an equation, but it is true for *all* values of the variable. We shall see shortly that $(x - 1)(x + 1) = x^2 - 1$ for any value of x whatsoever. This mean that the quantity on the left means exactly the same as that on the right whatever the value of x. To distinguish this usage from other uses of the equals symbol it is more correct to write $(x - 1)(x + 1) \equiv x^2 - 1$, where $\equiv$ means 'is identically equal to'. However, in practice, the equals sign is often used. We will only use $\equiv$ where it is particularly important to do so.

The 'not equals' sign ($\neq$)

The sign $\neq$ means 'is not equal to'. For example, $5 \neq 6$, $7 \neq -7$.

The notation for the change in a variable (δ)

The **change** in the value of a quantity is found by subtracting its initial value from its final value. For example, if the temperature of a mixture is initially $13°$C and at a later time is found to be $17°$C, the change in temperature is $17 - 13 = 4°$C. The Greek letter δ is often used to indicate such a change. If x is a variable we write δx to stand for a change in the value of x. We sometimes refer to δx as an **increment** in x. For example if the value of x changes from 3 to 3.01 we could write $\delta x = 3.01 - 3 = 0.01$. It is important to note that this is **not** the product of δ and x, rather the whole symbol 'δx' means 'the increment in x'.

Sigma (or summation) notation ($\sum$)

This provides a concise and convenient way of writing long sums.
The sum

$$x_1 + x_2 + x_3 + x_4 + \ldots + x_{11} + x_{12}$$

is written using the capital Greek letter sigma, $\sum$, as

$$\sum_{k=1}^{12} x_k$$

The symbol $\sum$ stands for the sum of all the values of x_k as k ranges from 1 to 12. Note that the lower-most and upper-most values of k are written at the bottom and top of the sigma sign respectively.

Example 10

Write out explicitly what is meant by $\displaystyle\sum_{k=1}^{5} k^3$.

Solution

We must let k range from 1 to 5. $\displaystyle\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$

Task Express $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ concisely using sigma notation.

Each term has the form $\frac{1}{k}$ where k varies from 1 to 4. Write down the sum using the sigma notation:

Your solution

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=$

Answer

$\sum_{k=1}^{4}\frac{1}{k}$

Example 11

Write out explicitly (a) $\sum_{k=1}^{3} 1$, (b) $\sum_{k=0}^{4} 2$.

Solution

(a) Here k does not appear explicitly in the terms to be added. This means add the constant 1, three times.

$$\sum_{k=1}^{3} 1 = 1+1+1 = 3$$

In general $\sum_{k=1}^{n} 1 = n$.

(b) Here k starts at zero so there are $n+1$ terms where $n=4$:

$$\sum_{k=0}^{4} 2 = 2+2+2+2+2 = 10$$

Exercises

1. State the reciprocal of (a) x, (b) $\dfrac{1}{z}$, (c) xy, (d) $\dfrac{1}{xy}$, (e) $a+b$, (f) $\dfrac{2}{a+b}$

2. The pressure p in a reaction vessel changes from 35 pascals to 38 pascals. Write down the value of δp.

3. Express as simply as possible (a) $(-3)\times x\times(-2)\times y$, (b) $9\times x\times z\times(-5)$.

4. Simplify (a) $8(2y)$, (b) $17x(-2y)$, (c) $5x(8y)$, (d) $5x(-8y)$

5. What is the distinction between $5x(2y)$ and $5x-2y$?

6. The value of x is 100 ± 3. The value of y is 120 ± 5. Find the maximum and minimum values of

   (a) $x+y$, (b) xy, (c) $\dfrac{x}{y}$, (d) $\dfrac{y}{x}$.

7. Write out explicitly (a) $\displaystyle\sum_{i=1}^{n} f_i$, (b) $\displaystyle\sum_{i=1}^{n} f_i x_i$.

8. By writing out the terms explicitly show that $\displaystyle\sum_{k=1}^{5} 3k = 3\sum_{k=1}^{5} k$

9. Write out explicitly $\displaystyle\sum_{k=1}^{3} y(x_k)\delta x_k$.

Answers

1. (a) $\dfrac{1}{x}$, (b) z, (c) $\dfrac{1}{xy}$, (d) xy, (e) $\dfrac{1}{a+b}$, (f) $\dfrac{a+b}{2}$.

2. $\delta p = 3$ pascals.

3. (a) $6xy$, (b) $-45xz$

4. (a) $16y$, (b) $-34xy$, (c) $40xy$, (d) $-40xy$

5. $5x(2y) = 10xy$, $5x-2y$ cannot be simplified.

6. (a) max 228, min 212, (b) 12875, 11155, (c) 0.8957, 0.7760, (d) 1.2887, 1.1165

7. (a) $\displaystyle\sum_{i=1}^{n} f_i = f_1 + f_2 + \ldots + f_{n-1} + f_n$,

   (b) $\displaystyle\sum_{i=1}^{n} f_i x_i = f_1x_1 + f_2x_2 + \ldots + f_{n-1}x_{n-1} + f_nx_n$.

9. $y(x_1)\delta x_1 + y(x_2)\delta x_2 + y(x_3)\delta x_3$.