MSc Mas6002, Introductory Material

Block A

Introduction to Probability and Statistics
Hints on Exercises

1(b). $P(\text{at least 2 aces}) = 1 - P(\text{no aces}) - P(\text{one ace})$

1(c). $P(\text{same number of clubs and spades}) = P(\text{no clubs, no spades})$
$+ P(\text{one club, one spade}) + P(\text{two clubs, two spades})$

2(a). Look at Venn diagram to identify (non-overlapping) mutually exclusive events which together form $A \cup B$.

2(b). $P(A \cup B \cup C) = P((A \cup B) \cup C)$

3(a). $P[\bigcap_1^n A_i] = 1 - P[\bigcup_1^n A_i^c]$

4. Definition of conditional probability. Rewrite $P(B \cap A^c)$ as in q.2

5. Need to show $P(A \cap B^c) = P(A)P(B^c)$ etc. Brief explanation only of generalization.

6. Direct enumeration.

7. Look up f, then find F.

8. Note $\sum_{x=1}^{\infty} x(1-\theta)^{x-1} = -\frac{\mathrm{d}}{\mathrm{d}\theta} \sum_{x=1}^{\infty} (1-\theta)^x$ and the sum on the right is the S_∞ of a G.P.

9. Use the substitution $z = \frac{x-\mu}{\sigma}$.

10. Find $E(X)$ and $E(X^2)$ (don't use $E(X(X-1))$). Use $\Gamma(t) = (t-1)\Gamma(t-1)$.

11. Note $P(X > a + b | X > a) = \frac{P(X>a+b \cap X>a)}{P(X>a)} = \frac{P(X>a+b)}{P(X>a)}$ since $b \geq 0$.

12. Solve the differential equation $\mathrm{Var}(g(X)) = \text{constant}$.

13. Direct enumeration.

14. Direct enumeration.

15. Solve $\int_R f(x)\mathrm{d}x = 1$, i.e. $\int_0^\infty \int_0^{x_1} ke^{-x_1}\mathrm{d}x_2\mathrm{d}x_1$ to find k.
Remember conditional range to find marginal $f_{X_1}(x_1) = \int_{R_{X_2|x_1}} f_{X_1,X_2}(x_1, x_2)\mathrm{d}x_2$.

16. Use hint given in section 3.2.

17. Not uniquely invertible, so back to first principles. If $Y = X^2$, then $P(Y \leq y) = P(-\sqrt{y} \leq x \leq \sqrt{y})$. Rewrite resulting integral by substitution, recognise integrand, recall $\chi_1^2 = Ga(\frac{1}{2}, \frac{1}{2})$.

18. Use results of section 3.4 with $\mathbf{Y} = (u, v), \mathbf{X} = (X, Y)$.

    Show that $f_{X,Y}(x, y) = e^{-(x+y)}$ for $x \in (0, \infty), y \in (0, \infty)$ and that $R_{U,V}$ is given by $R_U = (0, \infty)$, $R_V = (-u, u)$ - explain why. Then show $J(u, v) = -1/2$ and that $f_{U,V} = \frac{1}{2}e^{-u}$ for $(u, v) \in R_{U,V}$.

    Then integrate out.

19. Use property (i), Section 3.5.

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

20. Use general theory to specify distribution of $\sum X_i$.

    Find distribution of $\bar{X} = \frac{\sum X_i}{n}$ by considering $P(\bar{X} \leq x) = P(\sum X_i \leq nx)$ (recognise form, don't integrate).

    Find $E(\frac{1}{\bar{X}})$ directly from definition and distribution of $\bar{X}$. You do not need to evaluate any integrals - recognise their similarity to standard distributions.

21. Calculate bias and efficiency and compare.

22. Directly from definition. Recall $\log(\prod_{i=1}^{n} X_i) = \sum_{i=1}^{n} (\log X_i)$.

23. Note only one observation taken (though this is based on outcome of n 'subexperiments') so likelihood is just equal to probability function:

$$L(x, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}.$$

24. An example where the data do not come from identical distributions. However, they are independent. So the joint density or likelihood is still just the product of the separate different exponential densities.

25. Work exactly as suggested. Set up likelihood using given normal distribution for each x_i and make the substitution $v = \sigma^2$ to ease differentiation.

26. Use given hint.

    Working for power calculations very similar (in this case). Use Neave 3.2 to evaluate numerically.

27. 6.4(iii) gives $\frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \sim t_{n-1}$ when μ_0 is true mean.

    Now write down and invert a simple probability statement as in Chapter 7. Neave 3.1 gives the necessary critical values.

28. Like 34 but create probability statement from the distribution statement given in the hint. Note that the distribution is not symmetrical so you will need to get 2 critical values from the tables (Neave 3.2). [NB The 'symmetry' referred to only refers to taking $\alpha/2$ in each tail of the distribution to define the critical values, not to the fact that the values chosen are symmetrical about zero (this is a consequence in symmetric distributions).]

29. The data will take the form of a single value which may be one of $0, 1, 2$

    We must partition $0, 1, 2$ into acceptance and critical regions, e.g. accept if $X = 0$, reject if $X = 1$ or $X = 2$.

    Find all 8 such partitions and their associated errors.

30. Directly from definition. Take care over directions of inequalities (depends on $\lambda_1 > \lambda_0$).

31. Directly from definition. 'Supremum' is equivalent to 'maximum' in most cases. The final form doesn't simplify far.

32. Find $-2 \log \Lambda$ from definition. Show this is χ_2, by showing its square root is $N(0, 1)$ (if H_0 true).