MSc Mas6002, Introductory Material

Block A

Introduction to Probability and Statistics

Exercises

1. Find the probabilities that a hand of four cards drawn at random from a standard pack of 52 playing cards will contain

   (a) four cards of the same suit;

   (b) at least two 'aces';

   (c) the same number of 'clubs' as 'spades'.

2. (a) Prove, using the probability axioms, that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad (1)$$

   (b) By using the result $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

   Write down the 'general addition law' for n events i.e. $P(A_1 \cup A_2 \cup \ldots \cup A_n)$.

   (c) Deduce from the addition law (1)

$$P(A \cup B) \leq P(A) + P(B).$$

   Hence show,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq P(A_1) + P(A_2) + \ldots + P(A_n).$$

3. (a) Prove, using the result

$$(A_1 \cap A_2 \cap \ldots \cap A_n)^c = A_1^c \cup A_2^c \cup A_2^c \cup \ldots \cup A_n^c$$

   and the general addition law, Bonferroni's inequality:

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) \geq 1 - \sum_{i=1}^{n} P(A_i^c).$$

   (b) Suppose there are n events $A_1, A_2, \ldots, A_n$ and we require

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) \geq 1 - \alpha$$

   for some value α. What common value can be given to the individual probabilities $P(A_i)$ to ensure that this is true?

4. A and B are events with $0 < P(A) < 1$. Show that if $P(B|A) > P(B)$, then $P(B|A^c) < P(B)$.

5. If A and B are independent events, show that A and B^c are independent, and that A^c and B^c are independent. Discuss the generalization of this result to longer sequences of events.
6. A bag contains 5 white balls and 2 red balls. Balls are drawn at random one at a time without replacement until both red balls have been drawn. Find the probability function and distribution function of the number of draws required.
7. Sketch the distribution function $F(x)$ when
   i) $X \sim Ber(\frac{1}{2})$ (i.e. $X \sim Bi(1, \frac{1}{2})$),
   ii) $X \sim Bi(2, \frac{1}{2})$,
   iii) $X \sim Un(0, 1)$ (continuous).
8. The geometric distribution with parameter θ has probability function given by

$$p(x) = \begin{cases} \theta(1-\theta)^{x-1} & \text{if } x = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

   Find the mean, the second factorial moment, and hence the variance, of this distribution.
9. Verify, by direct consideration of the integrals involved, that if $X \sim N(\mu, \sigma^2)$

$$P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

   (see §2.4.3).
10. Verify that the mean and variance of the Gamma distribution are as given in §2.4.4
11. Show that the geometric distribution and the exponential distribution have the 'lack of memory' property: that is, if X is a random variable with one of these distributions, then $P(X > a + b | X > a)$ does not depend on a for $a, b > 0$.
12. If X is a random variable with $Po(\mu)$ distribution where μ is large, find a random variable $g(X)$ whose approximate variance (given by the first order Taylor approximation to g) does not depend on μ.
13. A r.v. X has p.d.f. $f(x) = x/8,\ 0 \le x \le 4$. Find the distribution of Y, the nearest integer to X. Compare the means and variances of X and Y.
14. Three counters numbered 1, 2 and 3 are placed in a container. Two are drawn at random one at a time without replacement. Let X and Y denote the numbers on the two counters respectively.
    (a) (Write down the joint probability function of (X, Y) and compute the probability that the number drawn first, X, is less than the number drawn second, Y.
    (b) Obtain the marginal distributions of X and Y and the conditional distribution $P(X = x \mid Y = y)$.
    (c) Are X and Y independent?
15. The joint p.d.f. of $\{X_1, X_2\}$ is $f(x_1, x_2) = ke^{-x_1}$ in the region $R = \{(x_1, x_2) : 0 < x_2 < x_1\}$ and zero elsewhere. Find k and the marginal p.d.f. of X_1.
16. Verify the relation

$$\text{Var}(X + Y) = \text{Var}\, X + \text{Var}\, Y + 2\,\text{Cov}(X, Y)$$

    given in §3.2.1.

17. If X has standard normal distribution, show that X^2 has χ_1^2 distribution. [Check: is this transformation uniquely invertible?]

18. If X, Y are i.i.d. $Ex(1)$ variables, write down the distribution of $X+Y$ and verify this by use of this transformation $U = X+Y, V = X-Y$.

19. Suppose $\mathbf{X} = (X_1, X_2, X_3)'$ is multivariate normal

$$\left\{ \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 & -2 & -2 \\ -2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix} \right\}.$$

Obtain the joint distribution of $Y_1 = X_1 + X_2, \quad Y_2 = X_1 + X_3$.

20. If $X_1, X_2, \ldots, X_n$ is a random sample from $Ex(\lambda)$, show that $\overline{X}$ has sampling distribution which is $Ga(n, n\lambda)$. Deduce that

$$E\left(\frac{1}{\overline{X}}\right) = \frac{n}{n-1}\lambda \qquad \text{for } n \geq 2.$$

21. Suppose that single observations X_1, X_2 are taken from a Poisson$(a\lambda)$ and a Poisson$(b\lambda)$ respectively (a, b are known positive constants and the observations are independent). Compare the following estimators of λ

$$\frac{X_1 + X_2}{a+b} \qquad \frac{X_1 - X_2}{a-b} \qquad \frac{1}{2}\left(\frac{X_1}{a} + \frac{X_2}{b}\right).$$

22. Find the maximum likelihood estimator of θ based on a random sample of size n from the distribution with density

$$f_X(x;\theta) = \theta x^{\theta-1} \qquad x \in (0,1), \theta > 0.$$

23. Let X be an observation from $Bi(n, \theta)$, where n is known and θ is unknown. Find the maximum likelihood estimator of θ, and show that it is unbiased.

24. Suppose X_i is the lifetime of patient i from time t_0 and z_i is his white blood cell count at t_0. A suitable model is thought to be

$$X_i \sim Ex(\lambda_i) \text{ where } \lambda_i = \beta z_i.$$

Data is available on n patients. Find the maximum likelihood estimate of β.

25. The normal linear regression model is given by

$$X_i \sim N(\alpha + \beta t_i, \sigma^2) \text{ for } i = 1, 2, \ldots, n$$

where $t_1, t_2, \ldots, t_n$ are fixed known quantitites, and α, β and σ^2 are unknown parameters.

Write down the log likelihood function. By differentiating it partially with respect to α, β and σ^2 (= v say), and setting the derivatives equal to zero, obtain expressions for the maximum likelihood estimators of α, β and σ^2.

26. $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with mean zero and unknown variance σ^2. Using the test statistic $\sum_{i=1}^n X_i^2$, suggest the form of a test of $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2$, where σ_0^2 and σ_1^2 are known with $\sigma_1^2 > \sigma_0^2$.
    Find a test of this form with size $\alpha = 0.05$.

    If $n = 25$, how large must σ_1^2/σ_0^2 be in order for this test to have power 0.95?

    Hint: use the result of q.17 and the fact that independent χ^2 variables are additive $(\chi_a^2 + \chi_b^2) = \chi_{a+b}^2$.

27. Construct a 90% confidence interval for μ using the information that a random sample of 16 observations from a $N(\mu, \sigma^2)$ distribution gave $\overline{x} = 95.8, s^2 = 30.25$.

    [Hint: use the t-distribution described towards the end of §5.5.3]

28. Construct a 95% confidence interval for σ^2 based on 40 observations from a $N(\mu, \sigma^2)$ distribution which gave $s^2 = 25$.

    Hint: use the result $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$.

29. A single observation is taken from the $Bi(2, \theta)$ distribution. Identify all eight possible tests of the hypotheses

$$H_0 : \theta = 1/2 \quad \text{v} \quad H_1 : \theta = 3/4$$

    and in each case give the type I and II error probabilities. Which test would you prefer?

30. Use the Neyman-Pearson lemma to provide a test of the hypothesis $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda = \lambda_1 > \lambda_0$ based on a random sample of size n from the Poisson (λ) distribution.

    [Hint: use the approximation $Po(\mu) \simeq N(\mu, \mu)$ when μ is large to evaluate the constant.]

31. Find the form of the likelihood ratio test of $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$ when $X_1, X_2, \ldots, X_n$ is a random sample from $Ex(\lambda)$. Simplify it as much as possible.

32. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where σ^2 is known. Find the form of the likelihood ratio test of $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Show that $-2 \log \Lambda$ is (exactly) χ_1^2 distributed if H_0 is true.

    [Hint: see hint for q.26.]