Contents 35

Sets and Probability

Learning outcomes

In this Workbook you will learn about probability. In the first Section you will learn about sets and how they may be combined together using the operations of union and intersection. Then you will learn how to apply the notation of sets to the notion of probability and learn about the fundamental laws of probability.

Sets

35.1

Introduction

If we can identify a property which is common to several objects, it is often useful to group them together. Such a grouping is called a **set**. Engineers for example, may wish to study all components of a production run which fail to meet some specified tolerance. Mathematicians may look at sets of numbers with particular properties, for example, the set of all even numbers, or the set of all numbers greater than zero. In this block we introduce some terminology that is commonly used to describe sets, and practice using set notation. This notation will be particularly useful when we come to study probability in Section 35.2.

Prerequisites

Before starting this Section you should ...

- have knowledge of basic algebra

Learning Outcomes

On completion you should be able to ...

- state what is meant by a set
- use set notation
- explain the concepts of the intersection and union of two sets
- define what is meant by the complement of a set
- use Venn diagrams to illustrate sets

1. Sets

A **set** is any collection of objects. Here, the word 'object' is used in its most general sense: an object may be a diode, an aircraft, a number, or a letter, for example.
A set is often described by listing the collection of objects - these are the members or **elements** of the set. We usually write this list of elements in curly brackets, and denote the full set by a capital letter. For example,

$C = \{$the resistors produced in a factory on a particular day$\}$
$D = \{$on, off$\}$
$E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

The elements of set C, above, are the resistors produced in a factory on a particular day. These could be individually labeled and listed individually but as the number is large it is not practical or sensible to do this. Set D lists the two possible states of a simple switch, and the elements of set E are the digits used in the decimal system.
Sometimes we can describe a set in words. For example,

'A is the set all odd numbers'.

Clearly all the elements of this set A cannot be listed.
Similarly,

'B is the set of binary digits' i.e. $B = \{0, 1\}$.

B has only two elements.

A set with a finite number of elements is called a **finite set**. B, C, D and E are finite sets. The set A has an infinite number of elements and so is not a finite set. It is called an **infinite set**.

Two sets are **equal** if they contain exactly the same elements. For example, the sets $\{9, 10, 14\}$ and $\{10, 14, 9\}$ are equal since the order in which elements are written is unimportant. Note also that repeated elements are ignored. The set $\{2, 3, 3, 3, 5, 5\}$ is equal to the set $\{2, 3, 5\}$.

Subsets

Sometimes one set is contained completely within another set. For example if $X = \{2, 3, 4, 5, 6\}$ and $Y = \{2, 3, 6\}$ then all the elements of Y are also elements of X. We say that Y is a **subset** of X and write $Y \subseteq X$.

Example 1

Given $A = \{0, 1, 2, 3\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$ and $C = \{0, 1\}$, state which sets are subsets of other sets.

Solution

A is a subset of B, that is $A \subseteq B$
C is a subset of B, that is $C \subseteq B$
C is a subset of A, that is $C \subseteq A$.

Task

A factory produces cars over a five day period; Monday to Friday. Consider the following sets,

(a) $A = \{$cars produced from Monday to Friday$\}$

(b) $B = \{$cars produced from Monday to Thursday$\}$

(c) $C = \{$cars produced on Friday$\}$

(d) $D = \{$cars produced on Wednesday$\}$

(e) $E = \{$cars produced on Wednesday or Thursday$\}$

State which sets are subsets of other sets.

Your solution

Answer

(a) B is a subset of A, that is, $B \subseteq A$.

(b) C is a subset of A, that is, $C \subseteq A$.

(c) D is a subset of A, that is, $D \subseteq A$.

(d) E is a subset of A, that is, $E \subseteq A$.

(e) D is a subset of B, that is, $D \subseteq B$.

(f) E is a subset of B, that is, $E \subseteq B$.

(g) D is a subset of E, that is, $D \subseteq E$.

The symbol $\in$

To show that an element belongs to a particular set we use the symbol $\in$. This symbol means **is a member of** or 'belongs to'. The symbol $\notin$ means **is not a member of** or 'does not belong to'.

For example if $X = \{$all even numbers$\}$ then we may write $4 \in X$, $6 \in X$, $7 \notin X$ and $11 \notin X$.

The empty set and the universal set

Sometimes a set will contain no elements. For example, suppose we define the set K by

$K = \{$all odd numbers which are divisible by 4$\}$

Since there are no odd numbers which are divisible by 4, then K has no elements. The set with no elements is called the **empty set**, and it is denoted by $\emptyset$.

On the other hand, the set containing all the objects of interest in a particular situation is called the **universal set**, denoted by S. The precise universal set will depend upon the context. If, for example, we are concerned only with whole numbers then $S = \{\cdots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$. If we are concerned only with the decimal digits then $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

The complement of a set

Given a set A and a universal set S we can define a new set, called the **complement** of A and denoted by A'. The complement of A contains all the elements of the universal set that are **not in** A.

Example 2

Given $A = \{2, 3, 7\}$, $B = \{0, 1, 2, 3, 4\}$ and $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ state
(a) A' (b) B'

Solution

(a) The elements of A' are those which belong to S but not to A.

$A' = \{0, 1, 4, 5, 6, 8, 9\}$

(b) $B' = \{5, 6, 7, 8, 9\}$

Sometimes a set is described in a mathematical way. Suppose the set Q contains all numbers which are divisible by 4 and 7. We can write

$Q = \{x : x$ is divisible by 4 and x is divisible by 7$\}$

The symbol : stands for 'such that '. We read the above as 'Q is the set comprising all elements x, such that x is divisible by 4 and by 7'.

2. Venn diagrams

Sets are often represented pictorially by **Venn diagrams** (see Figure 1).

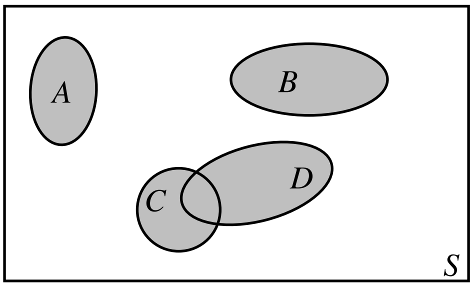


Figure 1

Here A, B, C, D represent sets. The sets A, B have no items in common so are drawn as non-intersecting regions whilst the sets C, D have some items in common so are drawn overlapping.
In a Venn diagram the universal set is represented by a rectangle and sets of interest by area regions within this rectangle.

Example 3

Represent the sets $A = \{0, 1\}$ and $B = \{0, 1, 2, 3, 4\}$ using a Venn diagram.

Solution

The elements 0 and 1 are in set A, represented by the small circle in the diagram. The large circle represents set B and so contains the elements 0,1,2,3 and 4. A suitable universal set in this case is the set of all integers. The universal set is shown by the rectangle.

Note that $A \subseteq B$. This is shown in the Venn diagram by A being completely inside B.

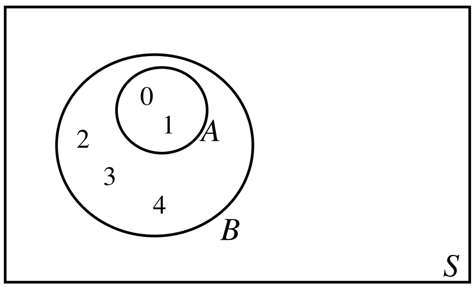


Figure 2: The set A is contained completely within B

Task

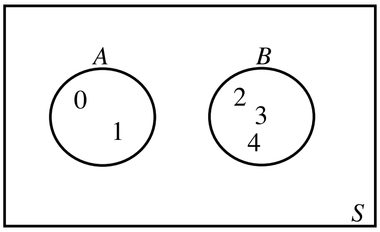
Given $A = \{0, 1\}$ and $B = \{2, 3, 4\}$ draw Venn diagrams showing
(a) A and B (b) A' (c) B'

Your solution

(a)

Answer

Note that A and B have no elements in common. This is represented pictorially in the Venn diagram by circles which are totally separate from each other as shown in the diagram.

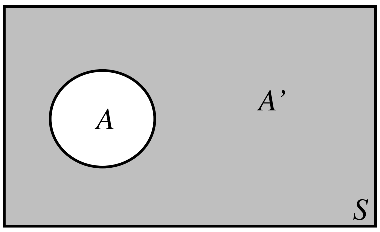


Your solution

(b)

Answer

The complement of A is the set whose elements do not belong to A. The set A' is shown shaded in the diagram.



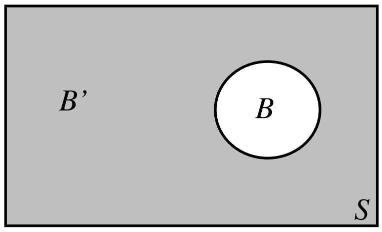
The complement of A contains elements which are not in A.

Your solution

(c)

Answer

The set B' is shown shaded in the diagram.



3. The intersection and union of sets

Intersection

Given two sets, A and B, the **intersection** of A and B is a set which contains elements that are common both to A and B. We write $A \cap B$ to denote the intersection of A and B. Mathematically we write this as:

Key Point 1

Intersection of Sets

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

This says that the intersection contains all the elements x such that x belongs to A and also x belongs to B.

Note that $A \cap B$ and $B \cap A$ are identical. The intersection of two sets can be represented by a Venn diagram as shown in Figure 3.

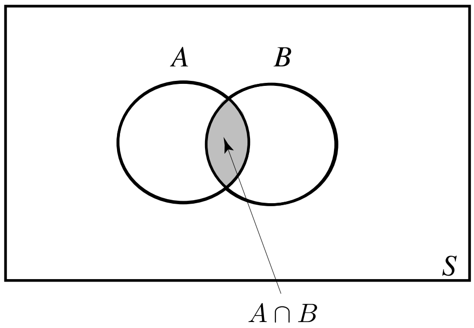


Figure 3: The overlapping area represents $A \cap B$

Example 4

Given $A = \{3, 4, 5, 6\}$, $B = \{3, 5, 9, 10, 15\}$ and $C = \{4, 6, 10\}$ state

(a) $A \cap B$, (b) $B \cap C$ and draw a Venn diagram representing these intersections.

Solution

(a) The elements common to both A and B are 3 and 5. Hence $A \cap B = \{3, 5\}$

(b) The only element common to B and C is 10. Hence $B \cap C = \{10\}$

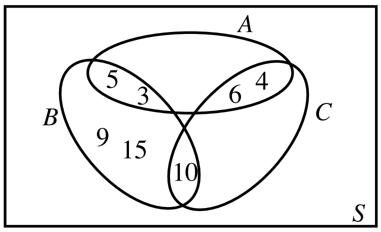


Figure 4

Task

Given $D = \{a, b, c\}$ and $F = \{\text{the entire alphabet}\}$ state $D \cap F$.

Your solution

Answer

The elements common to D and F are a, b and c, and so $D \cap F = \{a, b, c\}$

Note that D is a subset of F and so $D \cap F = D$.

The intersection of three or more sets is possible, and is the subject of the next Example.

Example 5

Given $A = \{0, 1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{2, 3, 4, 7, 9\}$ state

(a) $A \cap B$ (b) $(A \cap B) \cap C$ (c) $B \cap C$ (d) $A \cap (B \cap C)$

Solution

(a) The elements common to A and B are 1, 2 and 3 so $A \cap B = \{1, 2, 3\}$.

(b) We need to consider the sets $(A \cap B)$ and C. $A \cap B$ is given in (a). The elements common to $(A \cap B)$ and C are 2 and 3. Hence $(A \cap B) \cap C = \{2, 3\}$.

(c) The elements common to B and C are 2, 3 and 4 so $B \cap C = \{2, 3, 4\}$.

(d) We look at the sets A and $(B \cap C)$. The common elements are 2 and 3. Hence $A \cap (B \cap C) = \{2, 3\}$.

Note from (b) and (d) that here $(A \cap B) \cap C = A \cap (B \cap C)$.

The example illustrates a general rule. For any sets A, B and C it is true that

$$(A \cap B) \cap C = A \cap (B \cap C)$$

The position of the brackets is thus unimportant. They are usually omitted and we write $A \cap B \cap C$.

Suppose that sets A and B have no elements in common. Then their intersection contains no elements and we say that A and B are **disjoint** sets. We express this as

$$A \cap B = \emptyset$$

Recall that $\emptyset$ is the empty set. Disjoint sets are represented by separate area regions in the Venn diagram.

Union

The **union** of two sets A and B is a set which contains all the elements of A together with all the elements of B. We write $A \cup B$ to denote the union of A and B. We can describe the set $A \cup B$ formally by:

Key Point 2

Union of Sets

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}$$

Thus the elements of the set $A \cup B$ are those quantities x such that x is a member of A or a member of B or a member of both A and B. The deeply shaded areas of Figure 5 represents $A \cup B$.

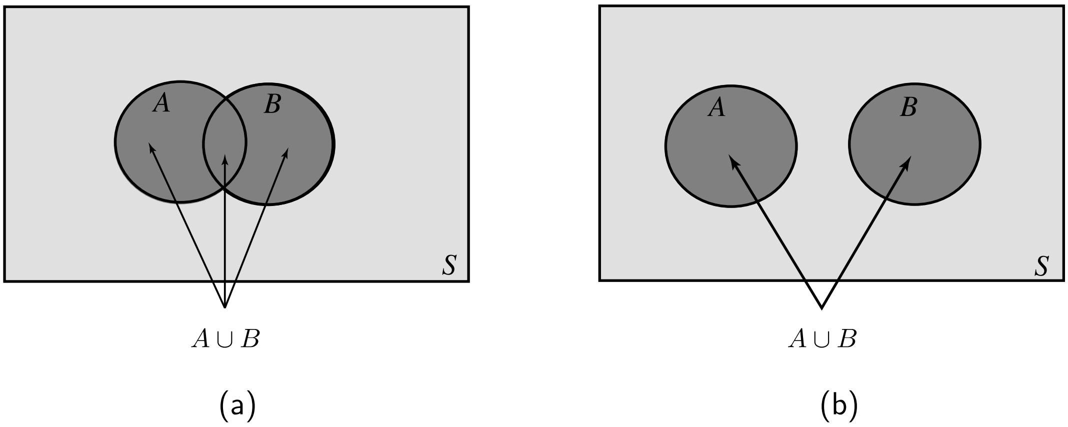


Figure 5

In Figure 5(a) the sets intersect, whereas in Figure 5(b) the sets have no region in common. We say they are **disjoint**.

Example 6

Given $A = \{0, 1\}$, $B = \{1, 2, 3\}$ and $C = \{2, 3, 4, 5\}$ write down

(a) $A \cup B$
(b) $A \cup C$
(c) $B \cup C$

Solution

(a) $A \cup B = \{0, 1, 2, 3\}$
(b) $A \cup C = \{0, 1, 2, 3, 4, 5\}$
(c) $B \cup C = \{1, 2, 3, 4, 5\}$.

Recall that there is no need to repeat elements in a set. Clearly the order of the union is unimportant so $A \cup B = B \cup A$.

Task

Given $A = \{2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{3, 5, 7, 9, 11\}$ state
(a) $A \cup B$ (b) $(A \cup B) \cap C$ (c) $A \cap B$ (d) $(A \cap B) \cup C$ (e) $A \cup B \cup C$

Your solution

Answer

(a) $A \cup B = \{2, 3, 4, 5, 6, 8, 10\}$

(b) We need to look at the sets $(A \cup B)$ and C. The elements common to both of these sets are 3 and 5. Hence $(A \cup B) \cap C = \{3, 5\}$.

(c) $A \cap B = \{2, 4, 6\}$

(d) We consider the sets $(A \cap B)$ and C. We form the union of these two sets to obtain $(A \cap B) \cup C = \{2, 3, 4, 5, 6, 7, 9, 11\}$.

(e) The set formed by the union of all three sets will contain all the elements from all the sets:

$$A \cup B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

Exercises

1. Given a set A, its complement A' and a universal set S, state which of the following expressions are true and which are false.

   (a) $A \cup A' = S$ (b) $A \cap S = \emptyset$ (c) $A \cap A' = \emptyset$
   (d) $A \cap A' = S$ (e) $A \cup \emptyset = S$ (f) $A \cup \emptyset = A$
   (g) $A \cup \emptyset = \emptyset$ (h) $A \cap \emptyset = A$ (i) $A \cap \emptyset = \emptyset$
   (j) $A \cup S = A$ (k) $A \cup S = \emptyset$ (l) $A \cup S = S$

2. Given $A = \{a, b, c, d, e, f\}$, $B = \{a, c, d, f, h\}$ and $C = \{e, f, x, y\}$ obtain the sets:

   (a) $A \cup B$ (b) $B \cap C$ (c) $A \cap (B \cup C)$
   (d) $C \cap (B \cup A)$ (e) $A \cap B \cap C$ (f) $B \cup (A \cap C)$

3. List the elements of the following sets:

   (a) $A = \{x : x$ is odd and x is greater than 0 and less than 12$\}$

   (b) $B = \{x : x$ is even and x is greater than 19 and less than 31$\}$

4. Given $A = \{5, 6, 7, 9\}$, $B = \{0, 2, 4, 6, 8\}$ and $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ list the elements of each of the following sets:

   (a) A' (b) B' (c) $A' \cup B'$
   (d) $A' \cap B'$ (e) $A \cup B$ (f) $(A \cup B)'$
   (g) $(A \cap B)'$ (h) $(A' \cap B)'$ (i) $(B' \cup A)'$

   What do you notice about your answers to (c),(g)?

   What do you notice about your answers to (d),(f)?

5. Given that A and B are intersecting sets, i.e. are not disjoint, show on a Venn diagran the following sets

   (a) A' (b) B' (c) $A \cup B'$ (d) $A' \cup B'$ (e) $A' \cap B'$

Answers

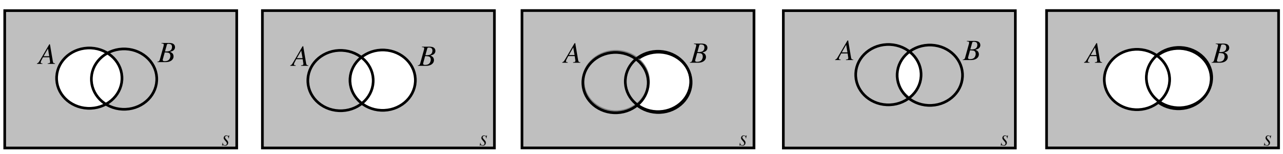
1.(a) T, (b) F, (c) T, (d) F, (e) F, (f) T, (g) F, (h) F, (i) T, (j) F, (k) F), (l) T.

2.(a) $\{a, b, c, d, e, f, h\}$, (b) $\{f\}$, (c) $\{a, c, d, e, f\}$, (d) $\{e, f\}$, (e) $\{f\}$, (f) $\{a, c, d, e, f, h\}$.

3.(a) $\{1, 3, 5, 7, 9, 11\}$, (b) $\{20, 22, 24, 26, 28, 30\}$.

4.(a) $\{0, 1, 2, 3, 4, 8\}$, (b) $\{1, 3, 5, 7, 9\}$, (c) $\{0, 1, 2, 3, 4, 5, 7, 8, 9\}$, (d) $\{1, 3\}$,
(e) $\{0, 2, 4, 5, 6, 7, 8, 9\}$, (f) $\{1, 3\}$, (g) $\{0, 1, 2, 3, 4, 5, 7, 8, 9\}$, (h) $\{1, 3, 5, 6, 7, 9\}$,
(i) $\{0, 2, 4, 8\}$.

5.



(a) (b) (c) (d) (e)