The Transform and its Inverse

20.2

Introduction

In this Section we formally introduce the Laplace transform. The transform is only applied to causal functions which were introduced in Section 20.1. We find the Laplace transform of many commonly occurring 'signals'and produce a table of standard Laplace transforms.
We also consider the inverse Laplace transform. To begin with, the inverse Laplace transform is obtained 'by inspection' using a table of transforms. This approach is developed by employing techniques such as partial fractions and completing the square introduced in HELM 3.6.

Prerequisites

Before starting this Section you should ...

- understand what a causal function is
- be able to find and use partial fractions
- be able to perform integration by parts
- be able to use the technique of completing the square

Learning Outcomes

On completion you should be able to ...

- find the Laplace transform of many commonly occurrring causal functions
- obtain the inverse Laplace transform using techniques involving
  (i) a table of transforms
  (ii) partial fractions
  (iii) completing the square
  (iv) the first shift theorem

1. The Laplace transform

If $f(t)$ is a **causal function** then the Laplace transform of $f(t)$ is written $\mathcal{L}\{f(t)\}$ and defined by:

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt.$$

Clearly, once the integral is performed and the limits substituted the resulting expression will involve the s parameter alone since the dependence upon t is removed in the integration process. This resulting expression in s is denoted by $F(s)$; its precise form is dependent upon the form taken by $f(t)$. We now refine Key Point 1 (page 4).

Key Point 3

The Laplace Transform of a Causal Function

$$\mathcal{L}\{f(t)u(t)\} \equiv \int_0^\infty \mathrm{e}^{-st} f(t)u(t)\, dt \equiv F(s)$$

To begin, we determine the Laplace transform of some simple causal functions. For example, if we consider the **ramp function** $f(t) = t.u(t)$ with graph

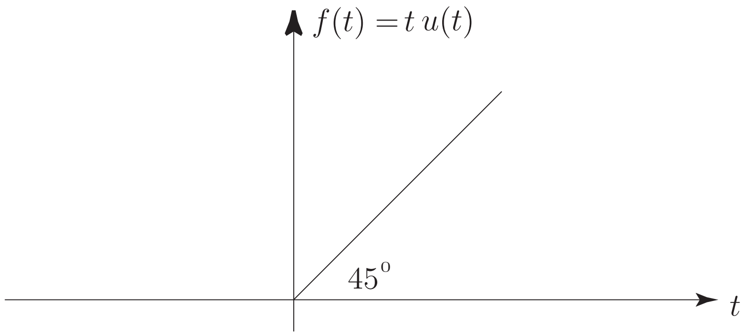

Figure 11

we find:

$$\begin{aligned}
\mathcal{L}\{t\ u(t)\} &= \int_0^\infty \mathrm{e}^{-st} t\ u(t)\, dt \\
&= \int_0^\infty \mathrm{e}^{-st} t\ \ dt \qquad \text{since in the range of the integral } u(t) = 1 \\
&= \left[\frac{t\ \mathrm{e}^{-st}}{(-s)}\right]_0^\infty - \int_0^\infty \frac{\mathrm{e}^{-st}}{(-s)}\ dt \qquad \text{using integration by parts} \\
&= \left[\frac{t\mathrm{e}^{-st}}{(-s)}\right]_0^\infty - \left[\frac{\mathrm{e}^{-st}}{(-s)^2}\right]_0^\infty
\end{aligned}$$

Now we have the difficulty of substituting in the limits of integration. The only problem arises with the upper limit ($t = \infty$). We shall always assume that the parameter s is so chosen that no contribution ever arises from the upper limit ($t = \infty$). In this particular case we need only demand that s is real and positive. Using this 'rule of thumb':

$$\begin{aligned}\mathcal{L}\{t\ u(t)\} &= [0-0] - \left[0 - \left(\frac{1}{(-s)^2}\right)\right] \\ &= \frac{1}{s^2}\end{aligned}$$

Thus, if $f(t) = t\ u(t)$ then $F(s) = 1/s^2$.
A similar, but more tedious, calculation yields the result that if $f(t) = t^n u(t)$ in which n is a positive integer then:

$$\mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}}$$

[We remember $n! \equiv n(n-1)(n-2)\ldots(3)(2)(1)$.]

Task

Find the Laplace transform of the step function $u(t)$.

Begin by obtaining the Laplace integral:

Your solution

Answer

You should obtain $\int_0^\infty \mathrm{e}^{-st}\,dt$ since in the range of integration, $t > 0$ and so $u(t) = 1$ leading to

$$\mathcal{L}\{u(t)\} = \int_0^\infty \mathrm{e}^{-st} u(t)\ dt = \int_0^\infty \mathrm{e}^{-st}\,dt$$

Your solution

Now complete the integration:

Answer

You should have obtained:

$$\begin{aligned}\mathcal{L}\{u(t)\} &= \int_0^\infty \mathrm{e}^{-st}\,dt \\ &= \left[\frac{\mathrm{e}^{-st}}{(-s)}\right]_0^\infty = 0 - \left[\frac{1}{(-s)}\right] = \frac{1}{s}\end{aligned}$$

where, again, we have assumed the contribution from the upper limit is zero.

As a second example, we consider the decaying exponential $f(t) = \mathrm{e}^{-at}u(t)$ where a is a positive constant. This function has graph:

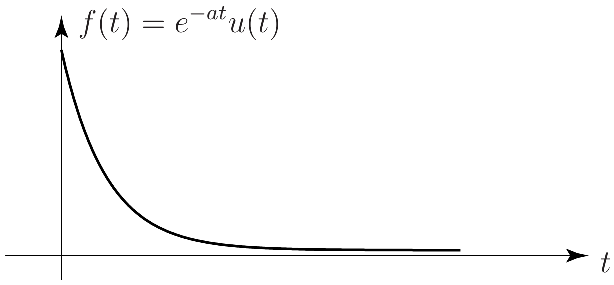

Figure 12

In this case,

$$\begin{aligned}
\mathcal{L}\{\mathrm{e}^{-at}u(t)\} &= \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-at}\,dt \\
&= \int_0^\infty \mathrm{e}^{-(s+a)t}\,dt \\
&= \left[\frac{\mathrm{e}^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a} \qquad \text{(zero contribution from the upper limit)}
\end{aligned}$$

Therefore, if $f(t) = \mathrm{e}^{-at}u(t)$ then $F(s) = \dfrac{1}{s+a}$.

Following this approach we can develop a table of Laplace transforms which records, for each causal function $f(t)$ listed, its corresponding transform function $F(s)$. Table 1 gives a limited table of transforms.

The linearity property of the Laplace transformation

If $f(t)$ and $g(t)$ are causal functions and c_1, c_2 are constants then

$$\begin{aligned}
\mathcal{L}\{c_1f(t) + c_2g(t)\} &= \int_0^\infty \mathrm{e}^{-st}[c_1f(t) + c_2g(t)]\,dt \\
&= c_1\int_0^\infty \mathrm{e}^{-st}f(t)\,dt + c_2\int_0^\infty \mathrm{e}^{-st}g(t)\,dt \\
&= c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}
\end{aligned}$$

Key Point 4

Linearity Property of the Laplace Transform

$$\mathcal{L}\{c_1f(t) + c_2g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}$$

Table 1. Table of Laplace Transforms

Rule	Causal function	Laplace transform
1	$f(t)$	$F(s)$
2	$u(t)$	$\dfrac{1}{s}$
3	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
4	$\mathrm{e}^{-at}u(t)$	$\dfrac{1}{s+a}$
5	$\sin at\,.\,u(t)$	$\dfrac{a}{s^2+a^2}$
6	$\cos at\,.\,u(t)$	$\dfrac{s}{s^2+a^2}$
7	$\mathrm{e}^{-at}\sin bt\,.\,u(t)$	$\dfrac{b}{(s+a)^2+b^2}$
8	$\mathrm{e}^{-at}\cos bt\,u(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$

Note: For convenience, this table is repeated at the end of the Workbook.

That is, the Laplace transform of a linear sum of causal functions is a linear sum of Laplace transforms. For example,

$$\begin{aligned}\mathcal{L}\{2\cos t\,.\,u(t) - 3t^2u(t)\} &= 2\mathcal{L}\{\cos t\,.\,u(t)\} - 3\mathcal{L}\{t^2u(t)\}\\ &= 2\left(\frac{s}{s^2+1}\right) - 3\left(\frac{2}{s^3}\right)\end{aligned}$$

Task

Obtain the Laplace transform of the hyperbolic function $\sinh at$.

Begin by expressing $\sinh at$ in terms of exponential functions:

Your solution

Answer

$\sinh at = \frac{1}{2}(\mathrm{e}^{at} - \mathrm{e}^{-at})$

Now use the linearity property (Key Point 4) to obtain the Laplace transform of the causal function $\sinh at.u(t)$:

Your solution

Answer

You should obtain $a/(s^2 - a^2)$ since

$$\begin{aligned}\mathcal{L}\{\sinh at.u(t)\} = \mathcal{L}\left\{\frac{\mathrm{e}^{at} - \mathrm{e}^{-at}}{2}.u(t)\right\} &= \frac{1}{2}\mathcal{L}\{\mathrm{e}^{at}.u(t)\} - \frac{1}{2}\mathcal{L}\{\mathrm{e}^{-at}.u(t)\} \\ &= \frac{1}{2}\left[\frac{1}{s-a}\right] - \frac{1}{2}\left[\frac{1}{s+a}\right] \qquad \text{(Table 1, Rule 4)} \\ &= \frac{1}{2}\left[\frac{2a}{(s-a)(s+a)}\right] = \frac{a}{s^2 - a^2}\end{aligned}$$

Task

Obtain the Laplace transform of the hyperbolic function $\cosh at$.

Your solution

Answer

You should obtain $\dfrac{s}{s^2 - a^2}$ since

$$\begin{aligned}\mathcal{L}\{\cosh at.u(t)\} = \mathcal{L}\left\{\frac{\mathrm{e}^{at} + \mathrm{e}^{-at}}{2}.u(t)\right\} &= \frac{1}{2}\mathcal{L}\{\mathrm{e}^{at}.u(t)\} + \frac{1}{2}\mathcal{L}\{\mathrm{e}^{-at}.u(t)\} \\ &= \frac{1}{2}\left[\frac{1}{s-a}\right] + \frac{1}{2}\left[\frac{1}{s+a}\right] \qquad \text{(Table 1, Rule 4)} \\ &= \frac{1}{2}\left[\frac{2s}{(s-a)(s+a)}\right] = \frac{s}{s^2 - a^2}\end{aligned}$$

Task

Find the Laplace transform of the **delayed step-function** $u(t-a)$, $a > 0$.

Write the delayed step-function here in terms of an integral:

Your solution

Answer

You should obtain $\mathcal{L}\{u(t-a)\} = \int_a^\infty \mathrm{e}^{-st}\,dt$ (note the lower limit is a) since:

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty \mathrm{e}^{-st}u(t-a)\,dt = \int_0^a \mathrm{e}^{-st}u(t-a)\,dt + \int_a^\infty \mathrm{e}^{-st}u(t-a)\,dt$$

In the first integral $0 < t < a$ and so $(t-a) < 0$, therefore $u(t-a) = 0$.

In the second integral $a < t < \infty$ and so $(t-a) > 0$, therefore $u(t-a) = 1$. Hence

$$\mathcal{L}\{u(t-a)\} = 0 + \int_a^\infty \mathrm{e}^{-st}\,dt.$$

Now complete the integration:

Your solution

Answer

$$\mathcal{L}\{u(t-a)\} = \int_a^\infty \mathrm{e}^{-st}\,dt = \left[\frac{\mathrm{e}^{-st}}{(-s)}\right]_a^\infty = \frac{\mathrm{e}^{-sa}}{s}$$

Exercise

Determine the Laplace transform of the following functions.

(a) $\mathrm{e}^{-3t}u(t)$ (b) $u(t-3)$ (c) $\mathrm{e}^{-t}\sin 3t.u(t)$ (d) $(5\cos 3t - 6t^3).u(t)$

Answer (a) $\dfrac{1}{s+3}$ (b) $\dfrac{\mathrm{e}^{-3s}}{s}$ (c) $\dfrac{3}{(s+1)^2+9}$ (d) $\dfrac{5s}{s^2+9} - \dfrac{36}{s^4}$

2. The inverse Laplace transform

The Laplace transform takes a causal function $f(t)$ and transforms it into a function of s, $F(s)$:

$$\mathcal{L}\{f(t)\} \equiv F(s)$$

The inverse Laplace transform operator is denoted by $\mathcal{L}^{-1}$ and involves recovering the original causal function $f(t)$. That is,

Key Point 5

Inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad \text{where} \quad \mathcal{L}\{f(t)\} = F(s)$$

For example, using standard transforms from Table 1:

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t \,.\, u(t) \text{ since } \mathcal{L}\{\cos 2t \,.\, u(t)\} = \frac{s}{s^2+4}. \qquad \text{(Table 1, Rule 6)}$$

Also

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3t\, u(t) \text{ since } \mathcal{L}\{3t\, u(t)\} = \frac{3}{s^2}. \qquad \text{(Table 1, Rule 3)}$$

Because the Laplace transform is a linear operator it follows that the inverse Laplace transform is also linear, so if $c_1,\ c_2$ are constants:

Key Point 6

Linearity Property of Inverse Laplace Transforms

$$\mathcal{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathcal{L}^{-1}\{F(s)\} + c_2\mathcal{L}^{-1}\{G(s)\}$$

For example, to find the inverse Laplace transform of $\dfrac{2}{s^4} - \dfrac{6}{s^2+4}$ we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{s^4} - \frac{6}{s^2+4}\right\} &= \frac{2}{6}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} - 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \frac{1}{3}t^3u(t) - 3\sin 2t \,.\, u(t) \qquad \text{(from Table 1)}\end{aligned}$$

Note that the fractions have had to be manipulated slightly in order that the expressions match precisely with the expressions in Table 1.

Although the inverse Laplace transform can be examined at a deeper mathematical level we shall be content with this simple-minded approach to finding inverse Laplace transforms by using the table of Laplace transforms. However, even this approach is not always straightforward and considerable algebraic manipulation is often required before an inverse Laplace transform can be found. Next we consider two standard rearrangements which often occur.

Inverting through the use of partial fractions

The function

$$F(s) = \frac{1}{(s-1)(s+2)}$$

does not appear in our table of transforms and so we cannot, by inspection, write down the inverse Laplace transform. However, by using partial fractions we see that

$$F(s) = \frac{1}{(s-1)(s+2)} = \frac{\frac{1}{3}}{s-1} - \frac{\frac{1}{3}}{s+2}$$

and so, using the linearity property:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s+2}\right\} \\ &= \tfrac{1}{3}\mathrm{e}^{t} - \tfrac{1}{3}\mathrm{e}^{-2t} \qquad \text{(Table 1, Rule 4)}\end{aligned}$$

Task

Find the inverse Laplace transform of $\dfrac{3}{(s-1)(s^2+1)}$.

Begin by using partial fractions to write the given expression in a more suitable form:

Your solution

Answer

$$\frac{3}{(s-1)(s^2+1)} = \frac{\frac{3}{2}}{s-1} - \frac{\frac{3}{2}s + \frac{3}{2}}{s^2+1}$$

Now continue to obtain the inverse:

Your solution

Answer

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{(s-1)(s^2+1)}\right\} &= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{3}{2}\left[\mathrm{e}^t - \cos t - \sin t\right] u(t) \qquad \text{(Table 1, Rules 4, 6, 5)}\end{aligned}$$

3. The first shift theorem

The first and second shift theorems enable an even wider range of Laplace transforms to be easily obtained than the transforms we have already found. They also enable a significantly wider range of inverse transforms to be found. Here we introduce the first shift theorem. If $f(t)$ is a causal function with Laplace transform $F(s)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$, then as we shall see, the Laplace transform of $\mathrm{e}^{-at}f(t)$, where a is a given constant, can easily be found in terms of $F(s)$.
Using the definition of the Laplace transform:

$$\begin{aligned}\mathcal{L}\{\mathrm{e}^{-at}f(t)\} &= \int_0^\infty \mathrm{e}^{-st}\left[\mathrm{e}^{-at}f(t)\right] dt \\ &= \int_0^\infty \mathrm{e}^{-(s+a)t} f(t)\, dt\end{aligned}$$

But if

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt$$

then simply replacing 's' by '$s+a$' on both sides gives:

$$F(s+a) = \int_0^\infty \mathrm{e}^{-(s+a)t} f(t)\, dt$$

That is, the parameter s is shifted to the value $s+a$.

We have then the statement of the **first shift theorem**:

Key Point 7

First Shift Theorem

If $\quad \mathcal{L}\{f(t)\} = F(s) \quad$ then $\quad \mathcal{L}\{\mathrm{e}^{-at}f(t)\} = F(s+a).$

For example, we already know (from Table 1) that

$$\mathcal{L}\{t^3u(t)\} = \frac{6}{s^4}$$

and so, by the first shift theorem:

$$\mathcal{L}\{\mathrm{e}^{-2t}t^3u(t)\} = \frac{6}{(s+2)^4}$$

Task

Use the first shift theorem to determine $\mathcal{L}\{\mathrm{e}^{2t}\cos 3t.u(t)\}$.

Your solution

Answer

You should obtain $\dfrac{s-2}{(s-2)^2+9}$ since $\mathcal{L}\{\cos 3t.u(t)\} = \dfrac{s}{s^2+9}$ (Table 1, Rule 6)

and so by the first shift theorem (with $a = -2$)

$$\mathcal{L}\{\mathrm{e}^{2t}\cos 3t.u(t)\} = \frac{s-2}{(s-2)^2+9}$$

obtained by simply replacing 's' by '$s-2$'.

We can also employ the first shift theorem to determine some inverse Laplace transforms.

Task

Find the inverse Laplace transform of $F(s) = \dfrac{3}{s^2-2s-8}$.

Begin by completing the square in the denominator:

Your solution

Answer

$$\frac{3}{s^2-2s-8} = \frac{3}{(s-1)^2-9}$$

Recalling that $\mathcal{L}\{\sinh 3t\ u(t)\} = \dfrac{3}{s^2-9}$ (from the Task on page 15) complete the inversion using the first shift theorem:

Your solution

Answer
You should obtain

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2-9}\right\} = \mathsf{e}^t \sinh 3t\ u(t)$$

Here, in the notation of the shift theorem:

$$f(t) = \sinh 3t\ u(t) \qquad F(s) = \frac{3}{s^2-9} \qquad \text{and} \quad a = -1$$

Inverting using completion of the square

The function:

$$F(s) = \frac{4s}{s^2+2s+5}$$

does not appear in the table of transforms and, again, needs amending before we can find its inverse transform. In this case, because s^2+2s+5 does not have nice factors, we complete the square in the denominator:

$$s^2+2s+5 \equiv (s+1)^2+4$$

and so

$$F(s) = \frac{4s}{s^2+2s+5} = \frac{4s}{(s+1)^2+4}$$

Now the numerator needs amending slightly to enable us to use the appropriate rule in the table of transforms (Table 1, Rule 8):

$$\begin{aligned} F(s) = \frac{4s}{(s+1)^2+4} &= 4\left\{\frac{s+1-1}{(s+1)^2+4}\right\} \\ &= 4\left\{\frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}\right\} \\ &= \frac{4(s+1)}{(s+1)^2+4} - 2\left[\frac{2}{(s+1)^2+4}\right] \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= 4\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\ &= 4\mathsf{e}^{-t}\cos 2t\,.\,u(t) - 2\mathsf{e}^{-t}\sin 2t\,.\,u(t) \\ &= \mathsf{e}^{-t}[4\cos 2t - 2\sin 2t]u(t) \end{aligned}$$

Task Find the inverse Laplace transform of $\dfrac{3}{s^2 - 4s + 6}$.

Begin by completing the square in the denominator of this expression:

Your solution

Answer

$$\frac{3}{s^2 - 4s + 6} = \frac{3}{(s-2)^2 + 2}$$

Now obtain the inverse:

Your solution

Answer

You should obtain:

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{\sqrt{2}}\left[\frac{\sqrt{2}}{(s-2)^2+2}\right]\right\} = \frac{3}{\sqrt{2}}\mathrm{e}^{2t}\sin\sqrt{2}t.u(t) \qquad \text{(Table 1, Rule 7)}$$

Exercise

Determine the inverse Laplace transforms of the following functions.

(a) $\dfrac{10}{s^4}$ (b) $\dfrac{s-1}{s^2+8s+17}$ (c) $\dfrac{3s-7}{s^2+9}$ (d) $\dfrac{3s+3}{(s-1)(s+2)}$ (e) $\dfrac{s+3}{s^2+4s}$

(f) $\dfrac{2}{(s+1)(s^2+1)}$

Answer

(a) $\frac{10}{6}t^3$ (b) $\mathrm{e}^{-4t}\cos t - 5\mathrm{e}^{-4t}\sin t$ (c) $3\cos 3t - \frac{7}{3}\sin 3t$ (d) $2\mathrm{e}^{t} + \mathrm{e}^{-2t}$

(e) $\frac{3}{4}u(t) + \frac{1}{4}\mathrm{e}^{-4t}u(t)$ (f) $(\mathrm{e}^{-t} - \cos t + \sin t)u(t)$