Even and Odd Functions

23.3

Introduction

In this Section we examine how to obtain Fourier series of periodic functions which are either *even* or *odd*. We show that the Fourier series for such functions is considerably easier to obtain as, if the signal is *even* only cosines are involved whereas if the signal is *odd* then only sines are involved. We also show that if a signal reverses after half a period then the Fourier series will only contain odd harmonics.

Prerequisites

Before starting this Section you should ...

- know how to obtain a Fourier series
- be able to integrate functions involving sinusoids
- have knowledge of integration by parts

Learning Outcomes

On completion you should be able to ...

- determine if a function is even or odd or neither
- easily calculate Fourier coefficients of even or odd functions

1. Even and odd functions

We have shown in the previous Section how to calculate, by integration, the coefficients a_n $(n = 0, 1, 2, 3, \ldots)$ and b_n $(n = 1, 2, 3, \ldots)$ in a Fourier series. Clearly this is a somewhat tedious process and it is advantageous if we can obtain as much information as possible without recourse to integration. In the previous Section we showed that the square wave (one period of which shown in Figure 12) has a Fourier series containing a constant term and cosine terms only (i.e. all the Fourier coefficients b_n are zero) while the function shown in Figure 13 has a more complicated Fourier series containing both cosine and sine terms as well as a constant.

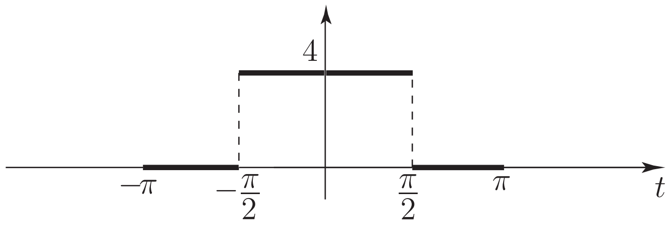


Figure 12: Square wave

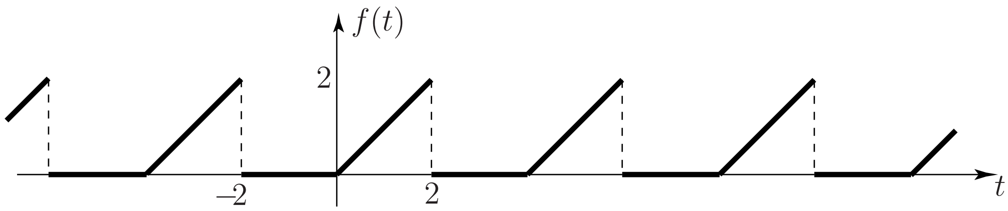


Figure 13: Saw-tooth wave

Task

Contrast the symmetry or otherwise of the functions in Figures 12 and 13.

Your solution

Answer
The square wave in Figure 12 has a graph which is symmetrical about the y-axis and is called an **even** function. The saw-tooth wave shown in Figure 13 has no particular symmetry.

In general a function is called **even** if its graph is unchanged under reflection in the y-axis. This is equivalent to

$$f(-t) = f(t) \qquad \text{for all } t$$

Obvious examples of even functions are $t^2, t^4, |t|, \cos t, \cos^2 t, \sin^2 t, \cos nt$.

A function is said to be **odd** if its graph is symmetrical about the origin (i.e. it has rotational symmetry about the origin). This is equivalent to the condition

$$f(-t) = -f(t)$$

Figure 14 shows an example of an odd function.

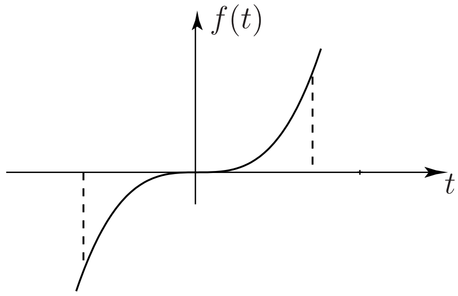


Figure 14

Examples of odd functions are $t, t^3, \sin t, \sin nt$. A periodic function which is odd is the saw-tooth wave in Figure 15.

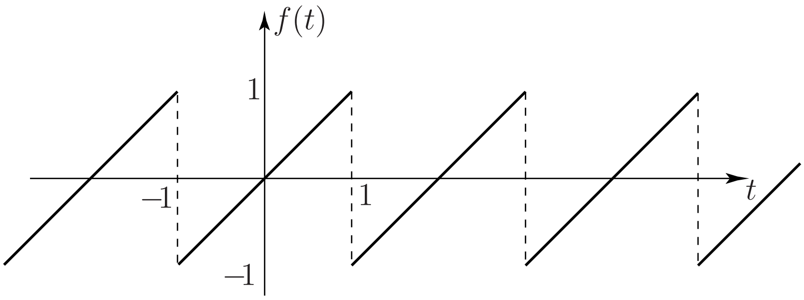
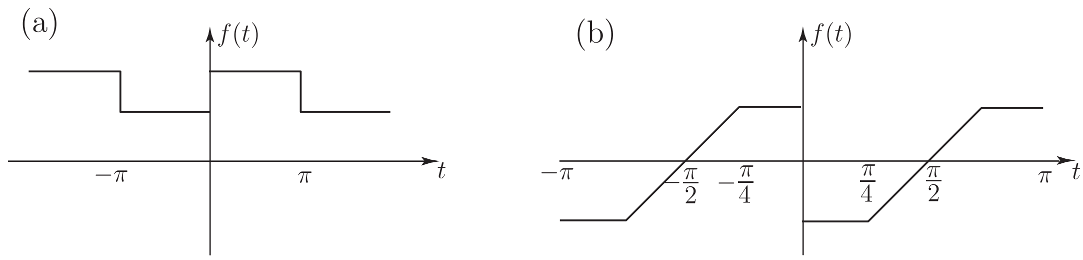


Figure 15

Some functions are **neither even nor odd**. The periodic saw-tooth wave of Figure 13 is an example; another is the exponential function e^t.

State the period of each of the following periodic functions and say whether it is even or odd or neither.



Your solution

Answer

(a) is neither even nor odd (with period 2π)

(b) is odd (with period π).

A Fourier series contains a *sum* of terms while the integral formulae for the Fourier coefficients a_n and b_n contain *products* of the type $f(t)\cos nt$ and $f(t)\sin nt$. We need therefore results for sums and products of functions.

Suppose, for example, $g(t)$ is an odd function and $h(t)$ is an even function.

$$\begin{aligned}
\text{Let} \quad F_1(t) &= g(t)\ h(t) && \text{(product of odd and even functions)}\\
\text{so} \quad F_1(-t) &= g(-t)h(-t) && \text{(replacing } t \text{ by } -t)\\
&= (-g(t))h(t) && \text{(since } g \text{ is odd and } h \text{ is even)}\\
&= -g(t)h(t) &&\\
&= -F_1(t) &&
\end{aligned}$$

So $F_1(t)$ is odd.

$$\begin{aligned}
\text{Now suppose} \qquad F_2(t) &= g(t) + h(t) && \text{(sum of odd and even functions)}\\
\therefore \qquad F_2(-t) &= g(-t) + h(t) &&\\
&= -g(t) + h(t) &&
\end{aligned}$$

$$\begin{aligned}
\text{We see that} \qquad F_2(-t) &\neq F_2(t)\\
\text{and} \qquad F_2(-t) &\neq -F_2(t)
\end{aligned}$$

So $F_2(t)$ is neither even nor odd.

Task

Investigate the odd/even nature of sums and products of

(a) two odd functions $g_1(t), g_2(t)$

(b) two even functions $h_1(t), h_2(t)$

Your solution

Answer

$$\begin{aligned} G_1(t) &= g_1(t)g_2(t) \\ G_1(-t) &= (-g_1(t))(-g_2(t)) \\ &= g_1(t)g_2(t) \\ &= G_1(t) \end{aligned}$$

so the product of two odd functions is even.

$$\begin{aligned} G_2(t) &= g_1(t) + g_2(t) \\ G_2(-t) &= g_1(-t) + g_2(-t) \\ &= -g_1(t) - g_2(t) \\ &= -G_2(t) \end{aligned}$$

so the sum of two odd functions is odd.

$$\begin{aligned} H_1(t) &= h_1(t)h_2(t) \\ H_2(t) &= h_1(t) + h_2(t) \end{aligned}$$

A similar approach shows that

$$\begin{aligned} H_1(-t) &= H_1(t) \\ H_2(-t) &= H_2(t) \end{aligned}$$

i.e. both the sum and product of two even functions are even.

These results are summarized in the following Key Point.

Key Point 5

Products of functions

$$\begin{aligned} \text{(even)} \times \text{(even)} &= \text{(even)} \\ \text{(even)} \times \text{(odd)} &= \text{(odd)} \\ \text{(odd)} \times \text{(odd)} &= \text{(even)} \end{aligned}$$

Sums of functions

$$\begin{aligned} \text{(even)} + \text{(even)} &= \text{(even)} \\ \text{(even)} + \text{(odd)} &= \text{(neither)} \\ \text{(odd)} + \text{(odd)} &= \text{(odd)} \end{aligned}$$

Useful properties of even and of odd functions in connection with integrals can be readily deduced if we recall that a definite integral has the significance of giving us the value of an area:

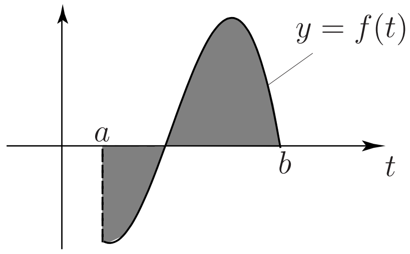


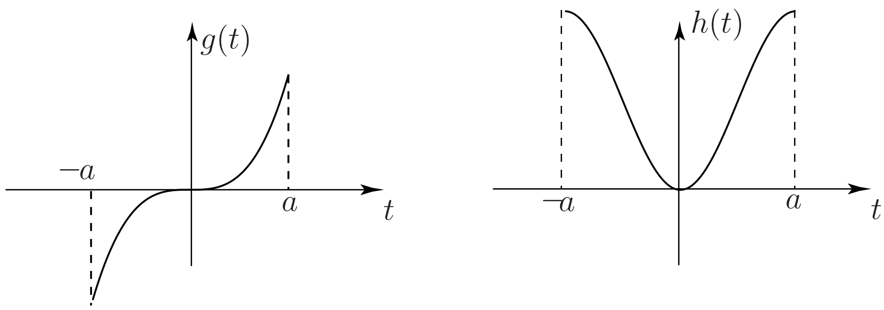
Figure 16

$\displaystyle\int_a^b f(t)\,dt$ gives us the **net** value of the shaded area, that above the t-axis being positive, that below being negative.

For the case of a symmetrical interval $(-a, a)$ deduce what you can about

$$\int_{-a}^{a} g(t)\,dt \qquad \text{and} \qquad \int_{-a}^{a} h(t)\,dt$$

where $g(t)$ is an odd function and $h(t)$ is an even function.



Your solution

Answer
We have

$$\int_{-a}^{a} g(t)\,dt = 0 \qquad \text{for an \textbf{odd} function}$$

$$\int_{-a}^{a} h(t)\,dt = 2\int_{0}^{a} h(t)\,dt \qquad \text{for an \textbf{even} function}$$

(Note that neither result holds for a function which is neither even nor odd.)

2. Fourier series implications

Since a sum of even functions is itself an even function it is not unreasonable to suggest that a Fourier series containing only cosine terms (and perhaps a constant term which can also be considered as an even function) can only represent an even periodic function. Similarly a series of sine terms (and no constant) can only represent an odd function. These results can readily be shown more formally using the expressions for the Fourier coefficients a_n and b_n.

Task

Recall that for a 2π-periodic function

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

If $f(t)$ is even, deduce whether the integrand is even or odd (or neither) and hence evaluate b_n. Repeat for the Fourier coefficients a_n.

Your solution

Answer

We have, if $f(t)$ is even,

$$f(t)\sin nt = (\text{even})\times(\text{odd}) = \text{odd}$$

hence $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}(\text{odd function})\ dt = 0$

Thus an even function has no sine terms in its Fourier series.

Also $f(t)\cos nt = (\text{even})\times(\text{even}) = \text{even}$

$$\therefore \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\text{even function})\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt.$$

It should be obvious that, for an odd function $f(t)$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}(\text{odd function})\,dt = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt$$

Analogous results hold for functions of any period, not necessarily 2π.

For a periodic function which is neither even nor odd we can expect at least some of both the a_n and b_n to be non-zero. For example consider the square wave function:

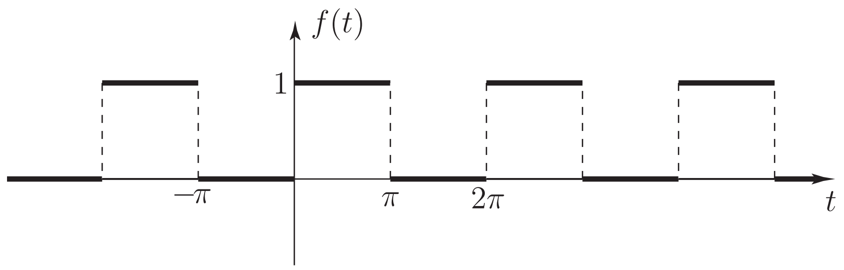


Figure 17: Square wave

This function is neither even nor odd and we have already seen in Section 23.2 that its Fourier series contains a constant $\left(\frac{1}{2}\right)$ and sine terms.

This result could be expected because we can write

$$f(t) = \frac{1}{2} + g(t)$$

where $g(t)$ is as shown:

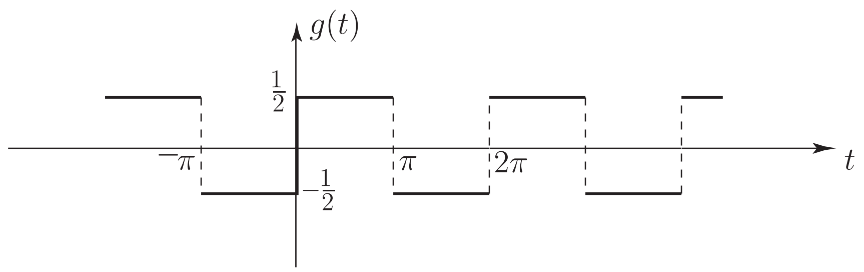


Figure 18

Clearly $g(t)$ is odd and will contain only sine terms. The Fourier series are in fact

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots\right)$$

and

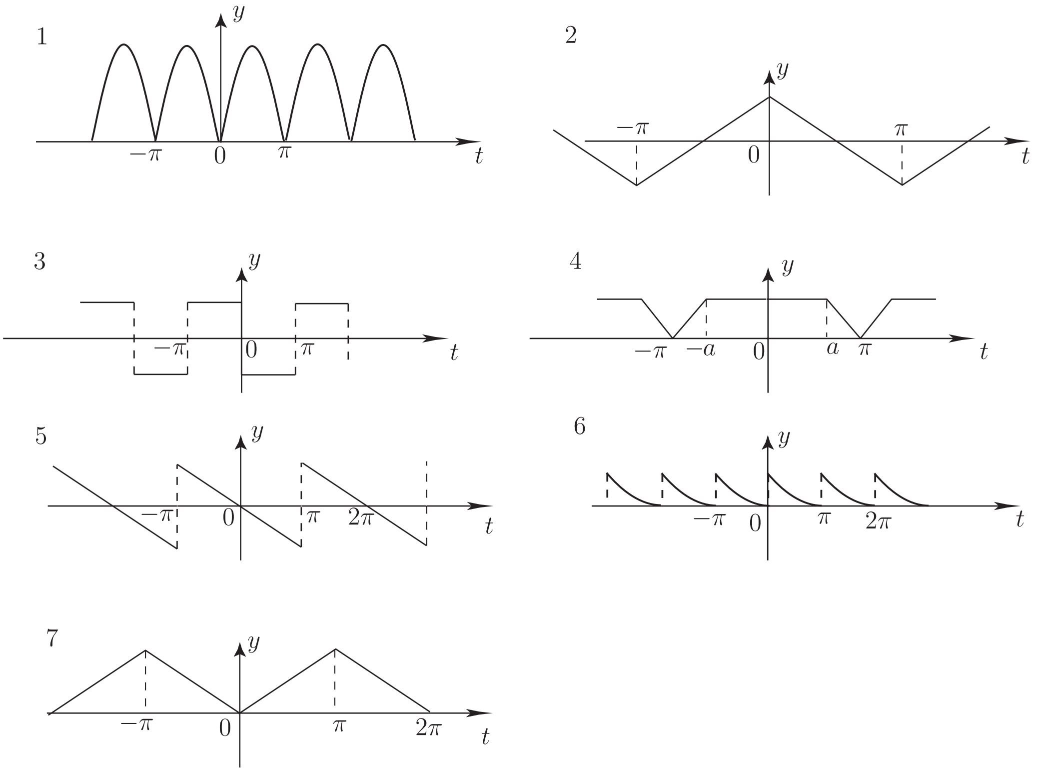
$$g(t) = \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots\right)$$

Task

For each of the following functions deduce whether the corresponding Fourier series contains

(a) sine terms only or cosine terms only or both

(b) a constant term



Your solution

Answer

1. cosine terms only (plus constant).
2. cosine terms only (no constant).
3. sine terms only (no constant).
4. cosine terms only (plus constant).
5. sine terms only (no constant).
6. sine and cosine terms (plus constant).
7. cosine terms only (plus constant).

Task

Confirm the result obtained for the triangular wave, function 7 in the last Task, by finding the Fourier series fully. The function involved is

$$\begin{aligned} f(t) &= |t| \qquad -\pi < t < \pi \\ f(t+2\pi) &= f(t) \end{aligned}$$

Your solution

Answer

Since $f(t)$ is even we can say immediately

$$b_n = 0 \qquad n = 1, 2, 3, \ldots$$

Also

$$a_n = \frac{2}{\pi}\int_0^{\pi} t\cos nt\,dt = \begin{cases} 0 & n \text{ even} \\ -\dfrac{4}{n^2\pi} & n \text{ odd} \end{cases} \qquad \text{(after integration by parts)}$$

Also $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} t\ dt = \pi$ so the Fourier series is

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \ldots\right)$$