Properties of the Fourier Transform

24.2

Introduction

In this Section we shall learn about some useful properties of the Fourier transform which enable us to calculate easily further transforms of functions and also in applications such as electronic communication theory.

Prerequisites

Before starting this Section you should ...

- be aware of the basic definitions of the Fourier transform and inverse Fourier transform

Learning Outcomes

On completion you should be able to ...

- state and use the linearity property and the time and frequency shift properties of Fourier transforms
- state various other properties of the Fourier transform

1. Linearity properties of the Fourier transform

(i) If $f(t)$, $g(t)$ are functions with transforms $F(\omega),\ G(\omega)$ respectively, then

- $\mathcal{F}\{f(t) + g(t)\} = F(\omega) + G(\omega)$

i.e. if we add 2 functions then the Fourier transform of the resulting function is simply the sum of the individual Fourier transforms.

(ii) If k is any constant,

- $\mathcal{F}\{kf(t)\} = kF(\omega)$

i.e. if we multiply a function by any constant then we must multiply the Fourier transform by the same constant. These properties follow from the definition of the Fourier transform and from the properties of integrals.

Examples

1.

$$\begin{aligned}
\mathcal{F}\{2e^{-t}u(t) + 3e^{-2t}u(t)\} &= \mathcal{F}\{2e^{-t}u(t)\} + \mathcal{F}\{3e^{-2t}u(t)\} \\
&= 2\mathcal{F}\{e^{-t}u(t)\} + 3\mathcal{F}\{e^{-2t}u(t)\} \\
&= \frac{2}{1+\mathrm{i}\omega} + \frac{3}{2+\mathrm{i}\omega}
\end{aligned}$$

2.

$$\begin{array}{rrcl}
\text{If} & f(t) &=& \begin{cases} 4 & -3 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases} \\
\text{then} & f(t) &=& 4p_3(t) \\
\text{so} & F(\omega) &=& 4P_3(\omega) = \dfrac{8}{\omega}\sin 3\omega
\end{array}$$

using the standard result for $\mathcal{F}\{p_a(t)\}$.

Task

If $f(t) = \begin{cases} 6 & -2 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$ write down $F(\omega)$.

Your solution

Answer

We have $f(t) = 6p_2(t)$ so $F(\omega) = \dfrac{12}{\omega}\sin 2\omega$.

2. Shift properties of the Fourier transform

There are two basic shift properties of the Fourier transform:

(i) Time shift property: • $\mathcal{F}\{f(t - t_0)\} = e^{-\mathrm{i}\omega t_0}F(\omega)$

(ii) Frequency shift property • $\mathcal{F}\{e^{\mathrm{i}\omega_0 t}f(t)\} = F(\omega - \omega_0)$.

Here t_0, ω_0 are constants.

In words, shifting (or translating) a function in one domain corresponds to a multiplication by a complex exponential function in the other domain.

We omit the proofs of these properties which follow from the definition of the Fourier transform.

Example 2

Use the time-shifting property to find the Fourier transform of the function

$$g(t) = \begin{cases} 1 & 3 \le t \le 5 \\ 0 & \text{otherwise} \end{cases}$$

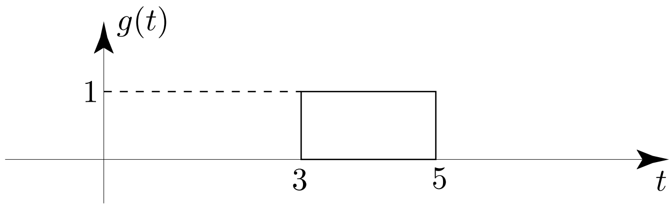


Figure 4

Solution

$g(t)$ is a pulse of width 2 and can be obtained by shifting the symmetrical rectangular pulse

$$p_1(t) = \begin{cases} 1 & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

by 4 units to the right.

Hence by putting $t_0 = 4$ in the time shift theorem

$$G(\omega) = \mathcal{F}\{g(t)\} = e^{-4\mathrm{i}\omega}\frac{2}{\omega}\sin\omega.$$

Task

Verify the result of Example 2 by direct integration.

Your solution

Answer

$$G(\omega) = \int_3^5 1e^{-\mathrm{i}\omega t}dt \quad = \quad \left[\frac{e^{-\mathrm{i}\omega t}}{-\mathrm{i}\omega}\right]_3^5 = \frac{e^{-5\mathrm{i}\omega} - e^{-3\mathrm{i}\omega}}{-\mathrm{i}\omega} = e^{-4\mathrm{i}\omega}\left(\frac{e^{\mathrm{i}\omega} - e^{-\mathrm{i}\omega}}{\mathrm{i}\omega}\right) = e^{-4\mathrm{i}\omega}2\frac{\sin\omega}{\omega},$$

as obtained using the time-shift property.

Task

Use the frequency shift property to obtain the Fourier transform of the modulated wave

$$g(t) = f(t)\cos\omega_0 t$$

where $f(t)$ is an arbitrary signal whose Fourier transform is $F(\omega)$.

First rewrite $g(t)$ in terms of complex exponentials:

Your solution

Answer

$$g(t) = f(t)\left(\frac{e^{\mathrm{i}\omega_0 t} + e^{-\mathrm{i}\omega_0 t}}{2}\right) = \frac{1}{2}f(t)e^{\mathrm{i}\omega_0 t} + \frac{1}{2}f(t)e^{-\mathrm{i}\omega_0 t}$$

Now use the linearity property and the frequency shift property on each term to obtain $G(\omega)$:

Your solution

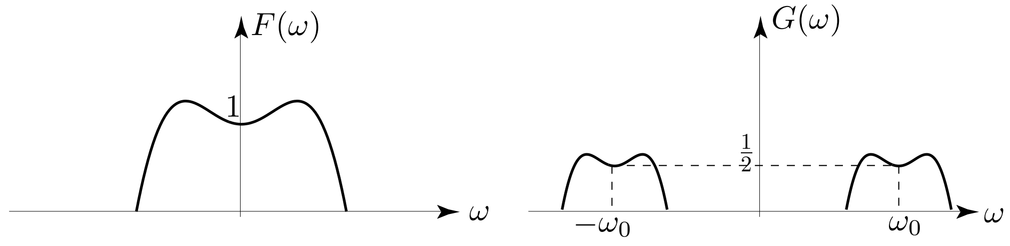
Answer

We have, by linearity:

$$\mathcal{F}\{g(t)\} = \frac{1}{2}\mathcal{F}\{f(t)e^{\mathrm{i}\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{f(t)e^{-\mathrm{i}\omega_0 t}\}$$

and by the frequency shift property:

$$G(\omega) = \frac{1}{2}F(\omega - \omega_0) + \frac{1}{2}F(\omega + \omega_0).$$



3. Inversion of the Fourier transform

Formal inversion of the Fourier transform, i.e. finding $f(t)$ for a given $F(\omega)$, is sometimes possible using the inversion integral (4). However, in elementary cases, we can use a Table of standard Fourier transforms together, if necessary, with the appropriate properties of the Fourier transform.

The following Examples and Tasks involve such inversion.

Example 3

Find the inverse Fourier transform of $F(\omega) = 20\dfrac{\sin 5\omega}{5\omega}$.

Solution

The appearance of the sine function implies that $f(t)$ is a symmetric rectangular pulse.

We know the standard form $\mathcal{F}\{p_a(t)\} = 2a\dfrac{\sin \omega a}{\omega a}$ or $\mathcal{F}^{-1}\{2a\dfrac{\sin \omega a}{\omega a}\} = p_a(t)$.

Putting $a = 5$ $\mathcal{F}^{-1}\{10\dfrac{\sin 5\omega}{5\omega}\} = p_5(t)$. Thus, by the linearity property

$$f(t) = \mathcal{F}^{-1}\{20\frac{\sin 5\omega}{5\omega}\} = 2p_5(t)$$

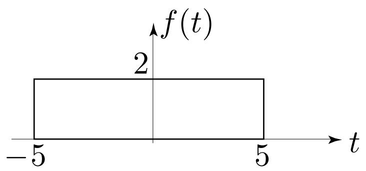


Figure 4

Example 4

Find the inverse Fourier transform of $G(\omega) = 20\dfrac{\sin 5\omega}{5\omega}\,\text{exp}\,(-3\mathrm{i}\omega)$.

Solution

The occurrence of the complex exponential factor in the Fourier transform suggests the time-shift property with the time shift $t_0 = +3$ (i.e. a right shift).

From Example 3

$$\mathcal{F}^{-1}\{20\frac{\sin 5\omega}{5\omega}\} = 2p_5(t) \qquad \text{so} \qquad g(t) = \mathcal{F}^{-1}\{20\frac{\sin 5\omega}{5\omega}e^{-3\mathrm{i}\omega}\} = 2p_5(t-3)$$

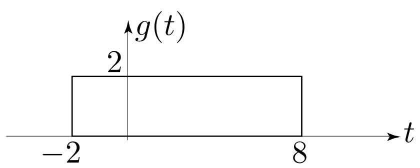


Figure 5

Task

Find the inverse Fourier transform of

$$H(\omega) = 6\frac{\sin 2\omega}{\omega}e^{-4i\omega}.$$

Firstly ignore the exponential factor and find the inverse Fourier transform of the remaining terms:

Your solution

Answer

We use the result: $\mathcal{F}^{-1}\{2a\dfrac{\sin \omega a}{\omega a}\} = p_a(t)$

Putting $a = 2$ gives $\mathcal{F}^{-1}\{2\dfrac{\sin 2\omega}{\omega}\} = p_2(t) \qquad \therefore \qquad \mathcal{F}^{-1}\{6\dfrac{\sin 2\omega}{\omega}\} = 3p_2(t)$

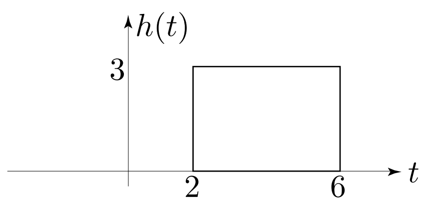
Now take account of the exponential factor:

Your solution

Answer

Using the time-shift theorem for $t_0 = 4$

$$h(t) = \mathcal{F}^{-1}\{6\frac{\sin 2\omega}{\omega}e^{-4i\omega}\} = 3p_2(t-4)$$



Example 5

Find the inverse Fourier transform of

$$K(\omega) = \frac{2}{1 + 2(\omega - 1)\mathrm{i}}$$

Solution

The presence of the term $(\omega - 1)$ instead of ω suggests the frequency shift property.

Hence, we consider first

$$\hat{K}(\omega) = \frac{2}{1 + 2\mathrm{i}\omega}.$$

The relevant standard form is

$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + \mathrm{i}\omega} \qquad \text{or} \qquad \mathcal{F}^{-1}\{\frac{1}{\alpha + \mathrm{i}\omega}\} = e^{-\alpha t}u(t).$$

Hence, writing $\hat{K}(\omega) = \dfrac{1}{\frac{1}{2} + \mathrm{i}\omega}$ $\qquad \hat{k}(t) = e^{-\frac{1}{2}t}u(t).$

Then, by the frequency shift property with $\omega_0 = 1$ $\qquad k(t) = \mathcal{F}^{-1}\{\dfrac{2}{1 + 2(\omega - 1)\mathrm{i}}\} = e^{-\frac{1}{2}t}e^{\mathrm{i}t}u(t).$

Here $k(t)$ is a complex time-domain signal.

Task

Find the inverse Fourier transforms of

(a) $L(\omega) = 2\dfrac{\sin\{3(\omega - 2\pi)\}}{(\omega - 2\pi)}$ $\qquad$ (b) $M(\omega) = \dfrac{e^{\mathrm{i}\omega}}{1 + \mathrm{i}\omega}$

Your solution

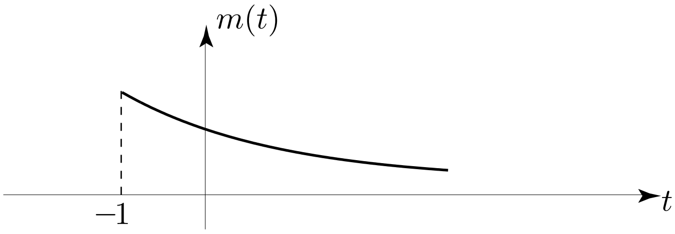
Answer

(a) Using the frequency shift property with $\omega_0 = 2\pi$

$l(t) = \mathcal{F}^{-1}\{L(\omega)\} = p_3(t)e^{\mathrm{i}2\pi t}$

(b) Using the time shift property with $t_0 = -1$

$m(t) = e^{-(t+1)}u(t+1)$



4. Further properties of the Fourier transform

We state these properties without proof. As usual $F(\omega)$ denotes the Fourier transform of $f(t)$.

(a) Time differentiation property:

$$\mathcal{F}\{f'(t)\} = \mathrm{i}\omega F(\omega)$$

(Differentiating a function is said to amplify the higher frequency components because of the additional multiplying factor ω.)

(b) Frequency differentiation property:

$$\mathcal{F}\{tf(t)\} = \mathrm{i}\frac{dF}{d\omega} \qquad \text{or} \qquad \mathcal{F}\{(-\mathrm{i}t)f(t)\} = \frac{dF}{d\omega}$$

Note the symmetry between properties (a) and (b).

(c) Duality property:

If $\mathcal{F}\{f(t)\} = F(\omega)$ then $\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$.

Informally, the duality property states that we can, apart from the 2π factor, interchange the time and frequency domains provided we put $-\omega$ rather than ω in the second term, this corresponding to a reflection in the vertical axis. If $f(t)$ is even this latter is irrelevant.

For example, we know that if $f(t) = p_1(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$, then $F(\omega) = 2\dfrac{\sin\omega}{\omega}$.

Then, by the duality property, since $p_1(\omega)$ is even, $\mathcal{F}\{2\dfrac{\sin t}{t}\} = 2\pi p_1(-\omega) = 2\pi p_1(\omega)$.

Graphically:

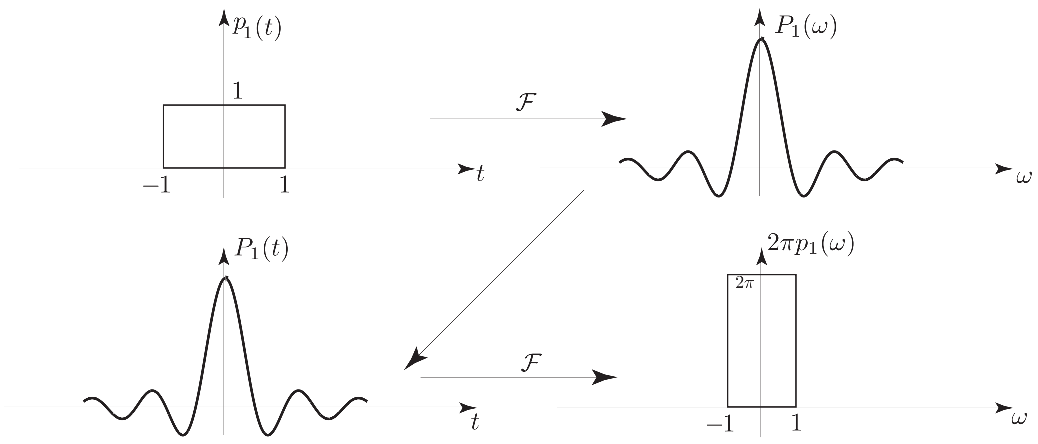


Figure 6

Task

Recalling the Fourier transform pair

$$f(t) = \begin{cases} e^{-2t} & t > 0 \\ e^{2t} & t < 0 \end{cases} \qquad F(\omega) = \frac{4}{4+\omega^2},$$

obtain the Fourier transforms of

(a) $g(t) = \dfrac{1}{4+t^2}$ (b) $h(t) = \dfrac{1}{4+t^2}\cos 2t.$

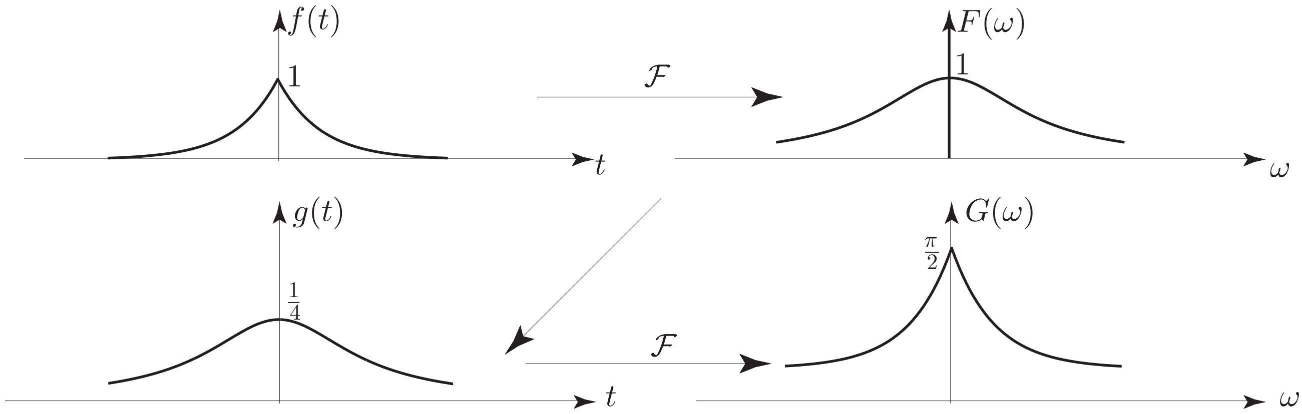
(a) Use the linearity and duality properties:

Your solution

Answer

We have $\mathcal{F}\{f(t)\} \equiv \mathcal{F}\{e^{-2|t|}\} = \dfrac{4}{4+\omega^2}$. $\quad \therefore \quad \mathcal{F}\{\dfrac{1}{4}e^{-2|t|}\} = \dfrac{1}{4+\omega^2}$ (by linearity)

$$\therefore \quad \mathcal{F}\{\frac{1}{4+t^2}\} = 2\pi\frac{1}{4}e^{-2|-\omega|} = \frac{\pi}{2}e^{-2|\omega|} = G(\omega) \qquad \text{(by duality).}$$



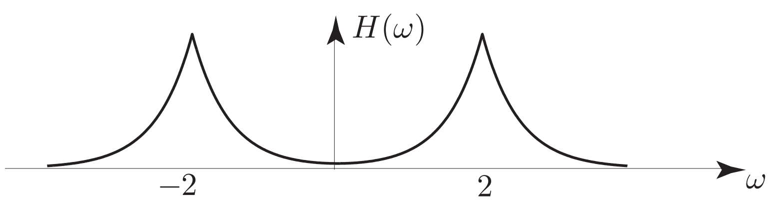
(b) Use the modulation property based on the frequency shift property:

Your solution

Answer

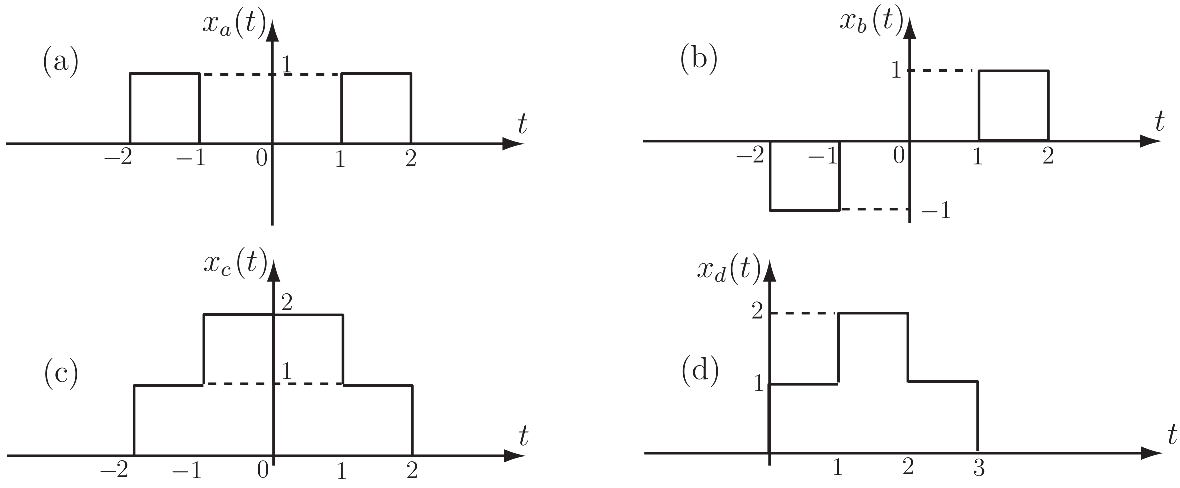
We have $h(t) = g(t)\cos 2t$. $\quad \therefore \quad \mathcal{F}\{g(t)\cos\omega_0 t\} = \frac{1}{2}(G(\omega-\omega_0) + G(\omega+\omega_0))$,

so with $\omega_0 = 2$ $\ \mathcal{F}\{h(t)\} = \dfrac{\pi}{4}\left\{e^{-2|\omega-2|} + e^{-2|\omega+2|}\right\} = H(\omega)$



Exercises

1. Using the superposition and time delay theorems and the known result for the transform of the rectangular pulse $p(t)$, obtain the Fourier transforms of each of the signals shown.



2. Obtain the Fourier transform of the signal

$$f(t) = \mathrm{e}^{-t}u(t) + \mathrm{e}^{-2t}u(t)$$

where $u(t)$ denotes the unit step function.

3. Use the time-shift property to obtain the Fourier transform of

$$f(t) = \begin{cases} 1 & 1 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Verify your result using the definition of the Fourier transform.

4. Find the inverse Fourier transforms of

(a) $F(\omega) = 20\dfrac{\sin(5\omega)}{5\omega}\mathrm{e}^{-3\mathrm{i}\omega}$

(b) $F(\omega) = \dfrac{8}{\omega}\sin 3\omega\ \mathrm{e}^{\mathrm{i}\omega}$

(c) $F(\omega) = \dfrac{\mathrm{e}^{\mathrm{i}\omega}}{1 - \mathrm{i}\omega}$

5. If $f(t)$ is a signal with transform $F(\omega)$ obtain the Fourier transform of $f(t)\cos(\omega_0 t)\cos(\omega_0 t)$.

Answer

1. $$X_a(\omega) = \frac{4}{\omega}\sin(\frac{\omega}{2})\cos(\frac{3\omega}{2})$$

$$X_b(\omega) = \frac{-4\mathrm{i}}{\omega}\sin(\frac{\omega}{2})\sin(\frac{3\omega}{2})$$

$$X_c(\omega) = \frac{2}{\omega}[\sin(2\omega) + \sin(\omega)]$$

$$X_d(\omega) = \frac{2}{\omega}\left(\sin(\frac{3\omega}{2}) + \sin(\frac{\omega}{2}\right)\mathrm{e}^{-3\mathrm{i}\omega/2}$$

2. $F(\omega) = \dfrac{3 + 2\mathrm{i}\omega}{2 - \omega^2 + 3\mathrm{i}\omega}$ (using the superposition property)

3. $F(\omega) = 2\dfrac{\sin\omega}{\omega}\mathrm{e}^{-2\mathrm{i}\omega}$

4. (a) $f(t) = \begin{cases} 2 & -2 < t < 8 \\ 0 & \text{otherwise} \end{cases}$

(b) $f(t) = \begin{cases} 4 & -4 < t < 2 \\ 0 & \text{otherwise} \end{cases}$

(c) $f(t) = \begin{cases} e^{t+1} & t < -1 \\ 0 & \text{otherwise} \end{cases}$

5. $\dfrac{1}{2}F(\omega) + \dfrac{1}{4}\left[F(\omega + 2\omega_0) + F(\omega - 2\omega_0)\right]$