Second Order Differential Equations 19.3

Introduction

In this Section we start to learn how to solve second order differential equations of a particular type: those that are linear and have constant coefficients. Such equations are used widely in the modelling of physical phenomena, for example, in the analysis of vibrating systems and the analysis of electrical circuits.

The solution of these equations is achieved in stages. The first stage is to find what is called a 'complementary function'. The second stage is to find a 'particular integral'. Finally, the complementary function and the particular integral are combined to form the general solution.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a differential equation
- understand complex numbers (HELM 10)

Learning Outcomes

On completion you should be able to ...

- recognise a linear, constant coefficient equation
- understand what is meant by the terms 'auxiliary equation' and 'complementary function'
- find the complementary function when the auxiliary equation has real, equal or complex roots

1. Constant coefficient second order linear ODEs

We now proceed to study those second order linear equations which have constant coefficients. The general form of such an equation is:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad (3)$$

where a, b, c are constants. The **homogeneous** form of (3) is the case when $f(x) \equiv 0$:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (4)$$

To find the general solution of (3), it is first necessary to solve (4). The general solution of (4) is called the **complementary function** and will always contain two arbitrary constants. We will denote this solution by y_{cf}.

The technique for finding the complementary function is described in this Section.

Task

State which of the following are constant coefficient equations.
State which are homogeneous.

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = \mathrm{e}^{-2x}$ (b) $x\dfrac{d^2y}{dx^2} + 2y = 0$

(c) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 7x = 0$ (d) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

Your solution

(a)

(b)

(c)

(d)

Answer

(a) is constant coefficient and is not homogeneous.

(b) is homogeneous but not constant coefficient as the coefficient of $\dfrac{d^2y}{dx^2}$ is x, a variable.

(c) is constant coefficient and homogeneous. In this example the dependent variable is x.

(d) is constant coefficient and homogeneous.

Note: A complementary function is the general solution of a homogeneous, linear differential equation.

2. Finding the complementary function

To find the complementary function we must make use of the following property.

If $y_1(x)$ and $y_2(x)$ are any two (linearly independent) solutions of a linear, homogeneous second order differential equation then the general solution $y_{\text{cf}}(x)$, is

$$y_{\text{cf}}(x) = Ay_1(x) + By_2(x)$$

where $A,\ B$ are constants.

We see that the second order linear ordinary differential equation has two arbitrary constants in its **general solution**. The functions $y_1(x)$ and $y_2(x)$ are **linearly independent** if one is not a multiple of the other.

Example 5

Verify that $y_1 = \mathrm{e}^{4x}$ and $y_2 = \mathrm{e}^{2x}$ both satisfy the constant coefficient linear homogeneous equation:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

Write down the general solution of this equation.

Solution

When $y_1 = \mathrm{e}^{4x}$, differentiation yields:

$$\frac{dy_1}{dx} = 4\mathrm{e}^{4x} \qquad \text{and} \qquad \frac{d^2y_1}{dx^2} = 16\mathrm{e}^{4x}$$

Substitution into the left-hand side of the ODE gives $16\mathrm{e}^{4x} - 6(4\mathrm{e}^{4x}) + 8\mathrm{e}^{4x}$, which equals 0, so that $y_1 = \mathrm{e}^{4x}$ is indeed a solution.

Similarly if $y_2 = \mathrm{e}^{2x}$, then

$$\frac{dy_2}{dx} = 2\mathrm{e}^{2x} \qquad \text{and} \qquad \frac{d^2y_2}{dx^2} = 4\mathrm{e}^{2x}.$$

Substitution into the left-hand side of the ODE gives $4\mathrm{e}^{2x} - 6(2\mathrm{e}^{2x}) + 8\mathrm{e}^{2x}$, which equals 0, so that $y_2 = \mathrm{e}^{2x}$ is also a solution of equation the ODE. Now e^{2x} and e^{4x} are linearly independent functions, so, from the property stated above we have:

$$y_{\text{cf}}(x) = A\mathrm{e}^{4x} + B\mathrm{e}^{2x} \quad \text{is the general solution of the ODE.}$$

Example 6

Find values of k so that $y = \mathrm{e}^{kx}$ is a solution of:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

Hence state the general solution.

Solution

As suggested we try a solution of the form $y = \mathrm{e}^{kx}$. Differentiating we find

$$\frac{dy}{dx} = k\mathrm{e}^{kx} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = k^2\mathrm{e}^{kx}.$$

Substitution into the given equation yields:

$$k^2\mathrm{e}^{kx} - k\mathrm{e}^{kx} - 6\mathrm{e}^{kx} = 0 \qquad \text{that is} \qquad (k^2 - k - 6)\mathrm{e}^{kx} = 0$$

The only way this equation can be satisfied for all values of x is if

$$k^2 - k - 6 = 0$$

that is, $(k-3)(k+2) = 0$ so that $k = 3$ or $k = -2$. That is to say, if $y = \mathrm{e}^{kx}$ is to be a solution of the differential equation, k must be either 3 or -2. We therefore have found two solutions:

$$y_1(x) = \mathrm{e}^{3x} \qquad \text{and} \qquad y_2(x) = \mathrm{e}^{-2x}$$

These are linearly independent and therefore the general solution is

$$y_{\text{cf}}(x) = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x}$$

The equation $k^2 - k - 6 = 0$ for determining k is called the **auxiliary equation**.

Task

By substituting $y = \mathrm{e}^{kx}$, find values of k so that y is a solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Hence, write down two solutions, and the general solution of this equation.

First find the auxiliary equation:

Your solution

Answer

$k^2 - 3k + 2 = 0$

Now solve the auxiliary equation and write down the general solution:

Your solution

Answer
The auxiliary equation can be factorised as $(k-1)(k-2)=0$ and so the required values of k are 1 and 2. The two solutions are $y=\text{e}^x$ and $y=\text{e}^{2x}$. The general solution is

$$y_{\text{cf}}(x)=A\text{e}^x+B\text{e}^{2x}$$

Example 7

Find the auxiliary equation of the differential equation:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

Solution

We try a solution of the form $y=\text{e}^{kx}$ so that

$$\frac{dy}{dx}=k\text{e}^{kx} \qquad \text{and} \qquad \frac{d^2y}{dx^2}=k^2\text{e}^{kx}.$$

Substitution into the given differential equation yields:

$$ak^2\text{e}^{kx}+bk\text{e}^{kx}+c\text{e}^{kx}=0 \qquad \text{that is} \qquad (ak^2+bk+c)\text{e}^{kx}=0$$

Since this equation is to be satisfied for all values of x, then

$$ak^2+bk+c=0$$

is the required auxiliary equation.

Key Point 5

The **auxiliary equation** of $\quad a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0 \quad$ is $\quad ak^2+bk+c=0\;$ where $\;y=\text{e}^{kx}$

Task

Write down, but do not solve, the auxiliary equations of the following:

(a) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0,$ (b) $2\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} - 3y = 0$

(c) $4\dfrac{d^2y}{dx^2} + 7y = 0,$ (d) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

Your solution

(a)

(b)

(c)

(d)

Answer

(a) $k^2 + k + 1 = 0$ (b) $2k^2 + 7k - 3 = 0$ (c) $4k^2 + 7 = 0$ (d) $k^2 + k = 0$

Solving the auxiliary equation gives the values of k which we need to find the complementary function. Clearly the nature of the roots will depend upon the values of $a,\ b$ and c.

Case 1 If $b^2 > 4ac$ the roots will be real and distinct. The two values of k thus obtained, k_1 and k_2, will allow us to write down two independent solutions: $y_1(x) = \mathrm{e}^{k_1 x}$ and $y_2(x) = \mathrm{e}^{k_2 x}$, and so the general solution of the differential equation will be:

$$y(x) = A\mathrm{e}^{k_1 x} + B\mathrm{e}^{k_2 x}$$

Key Point 6

If the auxiliary equation has real, distinct roots k_1 and k_2, the **complementary function** will be:

$$y_{\text{cf}}(x) = A\mathrm{e}^{k_1 x} + B\mathrm{e}^{k_2 x}$$

Case 2 On the other hand, if $b^2 = 4ac$ the two roots of the auxiliary equation will be equal and this method will therefore only yield one independent solution. In this case, special treatment is required.

Case 3 If $b^2 < 4ac$ the two roots of the auxiliary equation will be complex, that is, k_1 and k_2 will be complex numbers. The procedure for dealing with such cases will become apparent in the following examples.

Example 8

Find the general solution of: $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 10y = 0$

Solution

By letting $y = \mathrm{e}^{kx}$, so that $\dfrac{dy}{dx} = k\mathrm{e}^{kx}$ and $\dfrac{d^2y}{dx^2} = k^2\mathrm{e}^{kx}$

the auxiliary equation is found to be: $k^2 + 3k - 10 = 0$ and so $(k-2)(k+5) = 0$

so that $k = 2$ and $k = -5$. Thus there exist two solutions: $y_1 = \mathrm{e}^{2x}$ and $y_2 = \mathrm{e}^{-5x}$.

We can write the general solution as: $y = A\mathrm{e}^{2x} + B\mathrm{e}^{-5x}$

Example 9

Find the general solution of: $\dfrac{d^2y}{dx^2} + 4y = 0$

Solution

As before, let $y = \mathrm{e}^{kx}$ so that $\dfrac{dy}{dx} = k\mathrm{e}^{kx}$ and $\dfrac{d^2y}{dx^2} = k^2\mathrm{e}^{kx}$.

The auxiliary equation is easily found to be: $k^2 + 4 = 0$ that is, $k^2 = -4$ so that $k = \pm 2\mathrm{i}$, that is, we have complex roots. The two independent solutions of the equation are thus

$$y_1(x) = \mathrm{e}^{2\mathrm{i}x} \qquad y_2(x) = \mathrm{e}^{-2\mathrm{i}x}$$

so that the general solution can be written in the form $y(x) = A\mathrm{e}^{2\mathrm{i}x} + B\mathrm{e}^{-2\mathrm{i}x}$.

However, in cases such as this, it is usual to rewrite the solution in the following way.

Recall that Euler's relations give: $\mathrm{e}^{2\mathrm{i}x} = \cos 2x + \mathrm{i}\sin 2x$ and $\mathrm{e}^{-2\mathrm{i}x} = \cos 2x - \mathrm{i}\sin 2x$

so that $y(x) = A(\cos 2x + \mathrm{i}\sin 2x) + B(\cos 2x - \mathrm{i}\sin 2x)$.

If we now relabel the constants such that $A + B = C$ and $A\mathrm{i} - B\mathrm{i} = D$ we can write the general solution in the form:

$$y(x) = C\cos 2x + D\sin 2x$$

Note: In Example 8 we have expressed the solution as $y = \ldots$ whereas in Example 9 we have expressed it as $y(x) = \ldots$. Either will do.

Example 10

Given $ay'' + by' + cy = 0$, write down the auxiliary equation. If the roots of the auxiliary equation are complex (one root will always be the complex conjugate of the other) and are denoted by $k_1 = \alpha + \beta\text{i}$ and $k_2 = \alpha - \beta\text{i}$ show that the general solution is:

$$y(x) = \text{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Solution

Substitution of $y = \text{e}^{kx}$ into the differential equation yields $(ak^2 + bk + c)\text{e}^{kx} = 0$ and so the auxiliary equation is:

$$ak^2 + bk + c = 0$$

If $k_1 = \alpha + \beta\text{i},\ k_2 = \alpha - \beta\text{i}$ then the general solution is

$$y = C\text{e}^{(\alpha+\beta\text{i})x} + D\text{e}^{(\alpha-\beta\text{i})x}$$

where C and D are arbitrary constants.

Using the laws of indices this is rewritten as:

$$y = C\text{e}^{\alpha x}\text{e}^{\beta\text{i}x} + D\text{e}^{\alpha x}\text{e}^{-\beta\text{i}x} = \text{e}^{\alpha x}(C\text{e}^{\beta\text{i}x} + D\text{e}^{-\beta\text{i}x})$$

Then, using Euler's relations, we obtain:

$$\begin{aligned} y &= \text{e}^{\alpha x}(C\cos\beta x + C\text{i}\sin\beta x + D\cos\beta x - D\text{i}\sin\beta x) \\ &= \text{e}^{\alpha x}\{(C + D)\cos\beta x + (C\text{i} - D\text{i})\sin\beta x\} \end{aligned}$$

Writing $A = C + D$ and $B = C\text{i} - D\text{i}$, we find the required solution:

$$y = \text{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Key Point 7

If the auxiliary equation has complex roots, $\alpha + \beta\text{i}$ and $\alpha - \beta\text{i}$, then the **complementary function** is:

$$y_{\text{cf}} = \text{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Task Find the general solution of $y'' + 2y' + 4y = 0$.

Write down the auxiliary equation:

Your solution

Answer

$k^2 + 2k + 4 = 0$

Find the complex roots of the auxiliary equation:

Your solution

Answer

$k = -1 \pm \sqrt{3}\mathrm{i}$

Using Key Point 7 with $\alpha = -1$ and $\beta = \sqrt{3}$ write down the general solution:

Your solution

Answer

$y = \mathrm{e}^{-x}(A\cos\sqrt{3}x + B\sin\sqrt{3}x)$

Key Point 8

If the auxiliary equation has two equal roots, k, the **complementary function** is:

$$y_{\mathsf{cf}} = (A + Bx)\mathrm{e}^{kx}$$

Example 11

The auxiliary equation of $ay'' + by' + cy = 0$ is $ak^2 + bk + c = 0$. Suppose this equation has equal roots $k = k_1$ and $k = k_1$. Verify that $y = x\text{e}^{k_1 x}$ is a solution of the differential equation.

Solution

We have: $y = x\text{e}^{k_1x}$ $y' = \text{e}^{k_1x}(1 + k_1x)$ $y'' = \text{e}^{k_1x}(k_1^2x + 2k_1)$

Substitution into the left-hand side of the differential equation yields:

$$\text{e}^{k_1x}\{a(k_1^2x + 2k_1) + b(1 + k_1x) + cx\} = \text{e}^{k_1x}\{(ak_1^2 + bk_1 + c)x + 2ak_1 + b\}$$

But $ak_1^2 + bk_1 + c = 0$ since k_1 satisfies the auxiliary equation. Also,

$$k_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

but since the roots are equal, then $b^2 - 4ac = 0$ hence $k_1 = -b/2a$. So $2ak_1 + b = 0$. Hence $\text{e}^{k_1x}\{(ak_1^2 + bk_1 + c)x + 2ak_1 + b\} = \text{e}^{k_1x}\{(0)x + 0\} = 0$. We conclude that $y = x\text{e}^{k_1x}$ is a solution of $ay'' + by' + cy = 0$ when the roots of the auxiliary equation are equal. This illustrates Key Point 8.

Example 12

Obtain the general solution of the equation: $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0.$

Solution

As before, a trial solution of the form $y = \text{e}^{kx}$ yields an auxiliary equation $k^2 + 8k + 16 = 0$. This equation factorizes so that $(k + 4)(k + 4) = 0$ and we obtain equal roots, that is, $k = -4$ (twice). If we proceed as before, writing $y_1(x) = \text{e}^{-4x}$ $y_2(x) = \text{e}^{-4x}$, it is clear that the two solutions are not independent. We need to find a second independent solution. Using the result summarised in Key Point 8, we conclude that the second independent solution is $y_2 = x\text{e}^{-4x}$. The general solution is then:

$$y(x) = (A + Bx)\text{e}^{-4x}$$

Exercises

1. Obtain the general solutions, that is, the complementary functions, of the following equations:

(a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$ (b) $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 6y = 0$ (c) $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 6x = 0$

(d) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$ (e) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$ (f) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 8y = 0$

(g) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$ (h) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 5y = 0$ (i) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0$

(j) $\dfrac{d^2y}{dx^2} + 9y = 0$ (k) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 0$ (l) $\dfrac{d^2x}{dt^2} - 16x = 0$

2. Find the auxiliary equation for the differential equation $L\dfrac{d^2i}{dt^2} + R\dfrac{di}{dt} + \dfrac{1}{C}i = 0$

   Hence write down the complementary function.

3. Find the complementary function of the equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

Answers

1. (a) $y = A\mathrm{e}^{x} + B\mathrm{e}^{2x}$
   (b) $y = A\mathrm{e}^{-x} + B\mathrm{e}^{-6x}$
   (c) $x = A\mathrm{e}^{-2t} + B\mathrm{e}^{-3t}$
   (d) $y = A\mathrm{e}^{-t} + Bt\mathrm{e}^{-t}$
   (e) $y = A\mathrm{e}^{2x} + Bx\mathrm{e}^{2x}$
   (f) $y = \mathrm{e}^{-0.5t}(A\cos 2.78t + B\sin 2.78t)$
   (g) $y = A\mathrm{e}^{x} + Bx\mathrm{e}^{x}$
   (h) $x = \mathrm{e}^{-0.5t}(A\cos 2.18t + B\sin 2.18t)$
   (i) $y = A\mathrm{e}^{-2x} + B\mathrm{e}^{x}$
   (j) $y = A\cos 3x + B\sin 3x$
   (k) $y = A + B\mathrm{e}^{2x}$
   (l) $x = A\mathrm{e}^{4t} + B\mathrm{e}^{-4t}$

2. $Lk^2 + Rk + \dfrac{1}{C} = 0 \qquad i(t) = A\mathrm{e}^{k_1 t} + B\mathrm{e}^{k_2 t} \qquad k_1, k_2 = \dfrac{1}{2L}\left(-R \pm \sqrt{\dfrac{R^2C - 4L}{C}}\right)$

3. $\mathrm{e}^{-x/2}\left(A\cos \frac{\sqrt{3}}{2}x + B\sin \frac{\sqrt{3}}{2}x\right)$

3. The particular integral

Given a second order ODE

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = f(x),$$

a **particular integral** is any function, $y_{\mathsf{p}}(x)$, which satisfies the equation. That is, any function which when substituted into the left-hand side, results in the expression on the right-hand side.

Task

Show that

$$y = -\tfrac{1}{4}\mathrm{e}^{2x}$$

is a particular integral of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = \mathrm{e}^{2x} \qquad (1)$$

Starting with $y = -\frac{1}{4}\mathrm{e}^{2x}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$:

Your solution

Answer

$\dfrac{dy}{dx} = -\frac{1}{2}\mathrm{e}^{2x}, \quad \dfrac{d^2y}{dx^2} = -\mathrm{e}^{2x}$

Now substitute these into the ODE and simplify to check it satisfies the equation:

Your solution

Answer

Substitution yields $-\mathrm{e}^{2x} - \left(-\frac{1}{2}\mathrm{e}^{2x}\right) - 6\left(-\frac{1}{4}\mathrm{e}^{2x}\right)$ which simplifies to e^{2x}, the same as the right-hand side.

Therefore $y = -\frac{1}{4}\mathrm{e}^{2x}$ is a particular integral and we write (attaching a subscript p):

$$y_{\mathsf{p}}(x) = -\tfrac{1}{4}\mathrm{e}^{2x}$$

State what is meant by a particular integral.

Your solution

Answer

A particular integral is **any** solution of a differential equation.

4. Finding a particular integral

In the previous subsection we explained what is meant by a particular integral. Now we look at a simple method to find a particular integral. In fact our method is rather crude. It involves trial and error and educated guesswork. We try solutions which are of the same general form as the $f(x)$ on the right-hand side.

Example 13

Find a particular integral of the equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = \mathrm{e}^{2x}$$

Solution

We shall attempt to find a solution of the inhomogeneous problem by trying a function of the same form as that on the right-hand side of the ODE. In particular, let us try $y(x) = A\mathrm{e}^{2x}$, where A is a constant that we shall now determine. If $y(x) = A\mathrm{e}^{2x}$ then

$$\frac{dy}{dx} = 2A\mathrm{e}^{2x} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 4A\mathrm{e}^{2x}.$$

Substitution in the ODE gives:

$$4A\mathrm{e}^{2x} - 2A\mathrm{e}^{2x} - 6A\mathrm{e}^{2x} = \mathrm{e}^{2x}$$

that is,

$$-4A\mathrm{e}^{2x} = \mathrm{e}^{2x}$$

To ensure that y is a solution, we require $-4A = 1$, that is, $A = -\frac{1}{4}$.

Therefore the particular integral is $y_{\mathrm{p}}(x) = -\frac{1}{4}\mathrm{e}^{2x}$.

In Example 13 we chose a trial solution $A\mathrm{e}^{2x}$ of the same form as the ODE's right-hand side. Table 2 provides a summary of the trial solutions which should be tried for various forms of the right-hand side.

Table 2: Trial solutions to find the particular integral

	$f(x)$	Trial solution
(1)	constant term c	constant term k
(2)	linear, $ax + b$	$Ax + B$
(3)	polynomial in x of degree r: $ax^r + \cdots + bx + c$	polynomial in x of degree r: $Ax^r + \cdots + Bx + k$
(4)	$a \cos kx$	$A \cos kx + B \sin kx$
(5)	$a \sin kx$	$A \cos kx + B \sin kx$
(6)	$a\mathrm{e}^{kx}$	$A\mathrm{e}^{kx}$
(7)	$a\mathrm{e}^{-kx}$	$A\mathrm{e}^{-kx}$

Task

By trying a solution of the form $y = \alpha\mathrm{e}^{-x}$ find a particular integral of the equation
$\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 3\mathrm{e}^{-x}$

Substitute $y = \alpha\mathrm{e}^{-x}$ into the given equation to find α, and hence find the particular integral:

Your solution

Answer

$\alpha = -\frac{3}{2}$; $\quad y_{\mathsf{p}}(x) = -\frac{3}{2}\mathrm{e}^{-x}$

Example 14

Obtain a particular integral of the equation: $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = x.$

Solution

In Example 13 and the last Task, we found that a fruitful approach for a first order ODE was to assume a solution in the same form as that on the right-hand side. Suppose we assume a solution $y(x) = \alpha x$ and proceed to determine α. This approach will actually fail, but let us see why. If $y(x) = \alpha x$ then $\dfrac{dy}{dx} = \alpha$ and $\dfrac{d^2y}{dx^2} = 0$. Substitution into the differential equation yields $0 - 6\alpha + 8\alpha x = x$ and α.

Comparing coefficients of x:

$$8\alpha x = x \qquad \text{so} \quad \alpha = \frac{1}{8}$$

Comparing constants: $\qquad -6\alpha = 0 \qquad$ so $\quad \alpha = 0$

We have a contradiction! Clearly a particular integral of the form αx is not possible. The problem arises because differentiation of the term αx produces constant terms which are unbalanced on the right-hand side. So, we try a solution of the form $y(x) = \alpha x + \beta$ with α, β constants. This is consistent with the recommendation in Table 2 on page 43. Proceeding as before $\dfrac{dy}{dx} = \alpha$, $\dfrac{d^2y}{dx^2} = 0$. Substitution in the differential equation now gives:

$$0 - 6\alpha + 8(\alpha x + \beta) = x$$

Equating coefficients of x and then equating constant terms we find:

$$8\alpha = 1 \qquad (1)$$

$$-6\alpha + 8\beta = 0 \qquad (2)$$

From (1), $\alpha = \frac{1}{8}$ and then from (2) $\beta = \frac{3}{32}$.

The required particular integral is $y_{\mathsf{p}}(x) = \frac{1}{8}x + \frac{3}{32}$.

Task

Find a particular integral for the equation:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 3\cos x$$

First decide on an appropriate form for the trial solution, referring to Table 2 (page 43) if necessary:

Your solution

Answer

From Table 2, $y = A\cos x + B\sin x$, A and B constants.

Now find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and substitute into the differential equation:

Your solution

Answer

Differentiating, we find:

$$\frac{dy}{dx} = -A\sin x + B\cos x \qquad \frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

Substitution into the differential equation gives:

$$(-A\cos x - B\sin x) - 6(-A\sin x + B\cos x) + 8(A\cos x + B\sin x) = 3\cos x$$

Equate coefficients of $\cos x$:

Your solution

Answer

$7A - 6B = 3$

Also, equate coefficients of $\sin x$:

Your solution

Answer

$7B + 6A = 0$

Solve these two equations in A and B simultaneously to find values for A and B, and hence obtain the particular integral:

Your solution

Answer

$A = \frac{21}{85},\ B = -\frac{18}{85},\ y_{\mathsf{p}}(x) = \frac{21}{85}\cos x - \frac{18}{85}\sin x$

5. Finding the general solution of a second order linear inhomogeneous ODE

The general solution of a second order linear inhomogeneous equation is the sum of its particular integral and the complementary function. In subsection 2 (page 32) you learned how to find a complementary function, and in subsection 4 (page 42) you learnt how to find a particular integral. We now put these together to find the general solution.

Example 15

Find the general solution of $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 10y = 3x^2$

Solution

The complementary function was found in Example 8 to be $y_{\mathsf{cf}} = A\mathrm{e}^{2x} + B\mathrm{e}^{-5x}$.

The particular integral is found by trying a solution of the form $y = ax^2 + bx + c$, so that

$$\frac{dy}{dx} = 2ax + b, \qquad \frac{d^2y}{dx^2} = 2a$$

Substituting into the differential equation gives

$$2a + 3(2ax + b) - 10(ax^2 + bx + c) = 3x^2$$

Comparing constants: $2a + 3b - 10c = 0$

Comparing x terms: $6a - 10b = 0$

Comparing x^2 terms: $-10a = 3$

So $a = -\dfrac{3}{10}$, $b = -\dfrac{9}{50}$, $c = -\dfrac{57}{500}$, $\qquad y_{\mathsf{p}}(x) = -\dfrac{3}{10}x^2 - \dfrac{9}{50}x - \dfrac{57}{500}$.

Thus the general solution is $\quad y = y_{\mathsf{p}}(x) + y_{\mathsf{cf}}(x) = -\dfrac{3}{10}x^2 - \dfrac{9}{50}x - \dfrac{57}{500} + A\mathrm{e}^{2x} + B\mathrm{e}^{-5x}$

Key Point 9

The general solution of a second order constant coefficient ordinary differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad \text{is} \qquad y = y_{\mathsf{p}} + y_{\mathsf{cf}}$$

being the sum of the particular integral and the complementary function.

y_{p} contains no arbitrary constants; y_{cf} contains two arbitrary constants.

Engineering Example 2

An LC circuit with sinusoidal input

The differential equation governing the flow of current in a series LC circuit when subject to an applied voltage $v(t) = V_0 \sin \omega t$ is $L\dfrac{d^2i}{dt^2} + \dfrac{1}{C}i = \omega V_0 \cos \omega t$

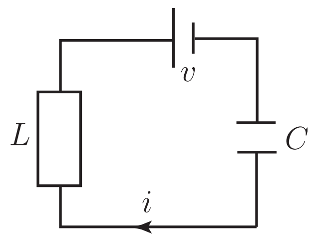

Figure 3

Obtain its general solution.

Solution

The homogeneous equation is $L\dfrac{d^2 i_{\mathsf{cf}}}{dt^2} + \dfrac{i_{\mathsf{cf}}}{C} = 0.$

Letting $i_{\mathsf{cf}} = \mathrm{e}^{kt}$ we find the auxiliary equation is $Lk^2 + \frac{1}{C} = 0$ so that $k = \pm \mathrm{i}/\sqrt{LC}$. Therefore, the complementary function is:

$$i_{\mathsf{cf}} = A\cos\frac{t}{\sqrt{LC}} + B\sin\frac{t}{\sqrt{LC}} \qquad \text{where } A \text{ and } B \text{ arbitrary constants.}$$

To find a particular integral try $i_{\mathsf{p}} = E\cos\omega t + F\sin\omega t$, where E, F are constants. We find:

$$\frac{di_{\mathsf{p}}}{dt} = -\omega E\sin\omega t + \omega F\cos\omega t \qquad \frac{d^2 i_{\mathsf{p}}}{dt^2} = -\omega^2 E\cos\omega t - \omega^2 F\sin\omega t$$

Substitution into the inhomogeneous equation yields:

$$L(-\omega^2 E\cos\omega t - \omega^2 F\sin\omega t) + \frac{1}{C}(E\cos\omega t + F\sin\omega t) = \omega V_0\cos\omega t$$

Equating coefficients of $\sin\omega t$ gives: $-\omega^2 LF + (F/C) = 0.$

Equating coefficients of $\cos\omega t$ gives: $-\omega^2 LE + (E/C) = \omega V_0.$

Therefore $F = 0$ and $E = CV_0\omega/(1 - \omega^2 LC)$. Hence the particular integral is

$$i_{\mathsf{p}} = \frac{CV_0\omega}{1 - \omega^2 LC}\cos\omega t.$$

Finally, the general solution is:

$$i = i_{\mathsf{cf}} + i_{\mathsf{p}} = A\cos\frac{t}{\sqrt{LC}} + B\sin\frac{t}{\sqrt{LC}} + \frac{CV_0\omega}{1 - \omega^2 LC}\cos\omega t$$

6. Inhomogeneous term in the complementary function

Occasionally you will come across a differential equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$ for which the inhomogeneous term, $f(x)$, forms part of the complementary function. One such example is the equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = \mathrm{e}^{3x}$$

It is straightforward to check that the complementary function is $y_{\mathsf{cf}} = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x}$. Note that the first of these terms has the same form as the inhomogeneous term, e^{3x}, on the right-hand side of the differential equation.

You should verify for yourself that trying a particular integral of the form $y_{\mathsf{p}}(x) = \alpha\mathrm{e}^{3x}$ will not work in a case like this. Can you see why?

Instead, try a particular integral of the form $y_{\mathsf{p}}(x) = \alpha x\mathrm{e}^{3x}$. Verify that

$$\frac{dy_{\mathsf{p}}}{dx} = \alpha\mathrm{e}^{3x}(3x+1) \qquad \text{and} \qquad \frac{d^2y_{\mathsf{p}}}{dx^2} = \alpha\mathrm{e}^{3x}(9x+6).$$

Substitute these expressions into the differential equation to find $\alpha = \frac{1}{5}$.

Finally, the particular integral is $y_{\mathsf{p}}(x) = \frac{1}{5}x\mathrm{e}^{3x}$ and so the general solution to the differential equation is:

$$y = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x} + \tfrac{1}{5}x\mathrm{e}^{3x}$$

This shows a generally effective method - where the inhomogeneous term $f(x)$ appears in the complementary function use as a trial particular integral x times what would otherwise be used.

Key Point 10

When solving

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

if the inhomogeneous term $f(x)$ appears in the complementary function use as a trial particular integral x times what would otherwise be used.

Exercises

1. Find the general solution of the following equations:

(a) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} - 3x = 6$ (b) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 8$ (c) $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 6y = 2t$

(d) $\dfrac{d^2x}{dt^2} + 11\dfrac{dx}{dt} + 30x = 8t$ (e) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 3y = 2\sin 2x$ (f) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 4\cos 3t$

(g) $\dfrac{d^2y}{dx^2} + 9y = 4\mathrm{e}^{8x}$ (h) $\dfrac{d^2x}{dt^2} - 16x = 9\mathrm{e}^{6t}$

2. Find a particular integral for the equation $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 5\mathrm{e}^{3t}$

3. Find a particular integral for the equation $\dfrac{d^2x}{dt^2} - x = 4\mathrm{e}^{-2t}$

4. Obtain the general solution of $y'' - y' - 2y = 6$

5. Obtain the general solution of the equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 10\cos 2x$

   Find the particular solution satisfying $y(0) = 1,\ \dfrac{dy}{dx}(0) = 0$

6. Find a particular integral for the equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 1 + x$

7. Find the general solution of

   (a) $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 5x = 3$ (b) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = \mathrm{e}^t$

Answers

1. (a) $x = A\mathrm{e}^{-t} + B\mathrm{e}^{3t} - 2$ (b) $y = A\mathrm{e}^{-x} + B\mathrm{e}^{-4x} + 2$ (c) $y = A\mathrm{e}^{-2t} + B\mathrm{e}^{-3t} + \frac{1}{3}t - \frac{5}{18}$

   (d) $x = A\mathrm{e}^{-6t} + B\mathrm{e}^{-5t} + 0.267t - 0.0978$

   (e) $y = \mathrm{e}^{-x}[A\sin\sqrt{2}x + B\cos\sqrt{2}x] - \frac{8}{17}\cos 2x - \frac{2}{17}\sin 2x$

   (f) $y = \mathrm{e}^{-0.5t}(A\cos 0.866t + B\sin 0.866t) - 0.438\cos 3t + 0.164\sin 3t$

   (g) $y = A\cos 3x + B\sin 3x + 0.0548\mathrm{e}^{8x}$ (h) $x = A\mathrm{e}^{4t} + B\mathrm{e}^{-4t} + \frac{9}{20}\mathrm{e}^{6t}$

2. $x_{\mathsf{p}} = 2.5\mathrm{e}^{3t}$

3. $x_{\mathsf{p}} = \frac{4}{3}\mathrm{e}^{-2t}$

4. $y = A\mathrm{e}^{2x} + B\mathrm{e}^{-x} - 3$

5. $y = A\mathrm{e}^{-2x} + B\mathrm{e}^{-x} + \frac{3}{2}\sin 2x - \frac{1}{2}\cos 2x$; $y = \frac{3}{2}\mathrm{e}^{-2x} + \frac{3}{2}\sin 2x - \frac{1}{2}\cos 2x$

6. $y_{\mathsf{p}} = x$

7. (a) $x = A\mathrm{e}^t + B\mathrm{e}^{5t} + \frac{3}{5}$ (b) $x = A\mathrm{e}^t + Bt\mathrm{e}^t + \frac{1}{2}t^2\mathrm{e}^t$