First Order Differential Equations 19.2

Introduction

Separation of variables is a technique commonly used to solve first order ordinary differential equations. It is so-called because we rearrange the equation to be solved such that all terms involving the dependent variable appear on one side of the equation, and all terms involving the independent variable appear on the other. Integration completes the solution. Not all first order equations can be rearranged in this way so this technique is not always appropriate. Further, it is not always possible to perform the integration even if the variables are separable.

In this Section you will learn how to decide whether the method is appropriate, and how to apply it in such cases.

An **exact** first order differential equation is one which can be solved by simply integrating both sides. Only very few first order differential equations are exact. You will learn how to recognise these and solve them. Some others may be converted simply to exact equations and that is also considered

Whilst exact differential equations are few and far between an important class of differential equations can be converted into exact equations by multiplying through by a function known as the **integrating factor** for the equation. In the last part of this Section you will learn how to decide whether an equation is capable of being transformed into an exact equation, how to determine the integrating factor, and how to obtain the solution of the original equation.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a differential equation; (Section 19.1)

Learning Outcomes

On completion you should be able to ...

- explain what is meant by separating the variables of a first order differential equation
- determine whether a first order differential equation is separable
- solve a variety of equations using the separation of variables technique

1. Separating the variables in first order ODEs

In this Section we consider differential equations which can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

Note that the right-hand side is a product of a function of x, and a function of y. Examples of such equations are

$$\frac{dy}{dx} = x^2\,y^3, \qquad \frac{dy}{dx} = y^2 \sin x \quad \text{and} \quad \frac{dy}{dx} = y \ln x$$

Not all first order equations can be written in this form. For example, it is not possible to rewrite the equation

$$\frac{dy}{dx} = x^2 + y^3$$

in the form

$$\frac{dy}{dx} = f(x)g(y)$$

Task

Determine which of the following differential equations can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

If possible, rewrite each equation in this form.

(a) $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$, (b) $\dfrac{dy}{dx} = 4x^2 + 2y^2$, (c) $y\dfrac{dy}{dx} + 3x = 7$

Your solution

Answer

(a) $\dfrac{dy}{dx} = x^2\left(\dfrac{1}{y^2}\right)$, (b) cannot be written in the stated form,

(c) Reformulating gives $\dfrac{dy}{dx} = (7 - 3x) \times \dfrac{1}{y}$ which is in the required form.

The variables involved in differential equations need not be x and y. Any symbols for variables may be used. Other first order differential equations are

$$\frac{dz}{dt} = t\mathrm{e}^z \qquad \frac{d\theta}{dt} = -\theta \quad \text{and} \quad \frac{dv}{dr} = v\left(\frac{1}{r^2}\right)$$

Given a differential equation in the form

$$\frac{dy}{dx} = f(x)g(y)$$

we can divide through by $g(y)$ to obtain

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

If we now integrate both sides of this equation with respect to x we obtain

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

that is

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

We have **separated the variables** because the left-hand side contains only the variable y, and the right-hand side contains only the variable x. We can now try to integrate each side separately. If we can actually perform the required integrations we will obtain a relationship between y and x. Examples of this process are given in the next subsection.

Key Point 1

Method of Separation of Variables

The solution of the equation

$$\frac{dy}{dx} = f(x)g(y)$$

may be found from separating the variables and integrating:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

2. Applying the method of separation of variables to ODEs

Example 3

Use the method of separation of variables to solve the differential equation

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

Solution

The equation already has the form

$$\frac{dy}{dx} = f(x)g(y)$$

where

$$f(x) = 3x^2 \qquad \text{and} \qquad g(y) = 1/y.$$

Dividing both sides by $g(y)$ we find

$$y\frac{dy}{dx} = 3x^2$$

Integrating both sides with respect to x gives

$$\int y\frac{dy}{dx}\,dx = \int 3x^2\,dx$$

that is

$$\int y\,dy = \int 3x^2\,dx$$

Note that the left-hand side is an integral involving just y; the right-hand side is an integral involving just x. After integrating both sides with respect to the stated variables we find

$$\tfrac{1}{2}y^2 = x^3 + c$$

where c is a constant of integration. (You might think that there would be a constant on the left-hand side too. You are quite right but the two constants can be combined into a single constant and so we need only write one.)

We now have a relationship between y and x as required. Often it is sufficient to leave your answer in this form but you may also be required to obtain an explicit relation for y in terms of x. In this particular case

$$y^2 = 2x^3 + 2c$$

so that

$$y = \pm\sqrt{2x^3 + 2c}$$

Task

Use the method of separation of variables to solve the differential equation

$$\frac{dy}{dx} = \frac{\cos x}{\sin 2y}$$

First separate the variables so that terms involving y and $\frac{dy}{dx}$ appear on the left, and terms involving x appear on the right:

Your solution

Answer

You should have obtained

$$\sin 2y \frac{dy}{dx} = \cos x$$

Now reformulate both sides as integrals:

Your solution

Answer

$$\int \sin 2y \frac{dy}{dx}\, dx = \int \cos x\, dx \quad \text{that is} \quad \int \sin 2y\, dy = \int \cos x\, dx$$

Now integrate both sides:

Your solution

Answer

$$-\tfrac{1}{2}\cos 2y = \sin x + c$$

Finally, rearrange to obtain an expression for y in terms of x:

Your solution

Answer

$y = \frac{1}{2}\cos^{-1}(D - 2\sin x)$ where $D = -2c$

Exercises

1. Solve the equation

$$\frac{dy}{dx} = \frac{\mathsf{e}^{-x}}{y}.$$

2. Solve the following equation subject to the condition $y(0) = 1$:

$$\frac{dy}{dx} = 3x^2\mathsf{e}^{-y}$$

3. Find the general solution of the following equations:

(a) $\dfrac{dy}{dx} = 3$, (b) $\dfrac{dy}{dx} = \dfrac{6\sin x}{y}$

4. (a) Find the general solution of the equation

$$\frac{dx}{dt} = t(x-2).$$

(b) Find the particular solution which satisfies the condition $x(0) = 5$.

5. Some equations which do not appear to be separable can be made so by means of a suitable substitution. By means of the substitution $z = y/x$ solve the equation

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x} + 1$$

6. The equation

$$iR + L\frac{di}{dt} = E$$

where R, L and E are constants arises in electrical circuit theory. This equation can be solved by separation of variables. Find the solution which satisfies the condition $i(0) = 0$.

Answers

1. $y = \pm\sqrt{D - 2\mathsf{e}^{-x}}$.

2. $y = \ln(x^3 + e)$.

3 (a) $y = 3x + C$, (b) $\frac{1}{2}y^2 = C - 6\cos x$.

4. (a) $x = 2 + A\mathsf{e}^{t^2/2}$, (b) $x = 2 + 3\mathsf{e}^{t^2/2}$.

5. $z = \tan(\ln Dx)$ so that $y = x\tan(\ln Dx)$.

6. $i = \dfrac{E}{R}(1 - \mathsf{e}^{-t/\tau})$ where $\tau = L/R$.

3. Exact equations

Consider the differential equation

$$\frac{dy}{dx} = 3x^2$$

By direct integration we find that the general solution of this equation is

$$y = x^3 + C$$

where C is, as usual, an arbitrary constant of integration.
Next, consider the differential equation

$$\frac{d}{dx}(yx) = 3x^2.$$

Again, by direct integration we find that the general solution is

$$yx = x^3 + C.$$

We now divide this equation by x to obtain

$$y = x^2 + \frac{C}{x}.$$

The differential equation $\dfrac{d}{dx}(yx) = 3x^2$ is called an **exact equation**. It can effectively be solved by integrating both sides.

Task

Solve the equations (a) $\dfrac{dy}{dx} = 5x^4$ (b) $\dfrac{d}{dx}(x^3y) = 5x^4$

Your solution

(a) $y =$ (b) $y =$

Answer

(a) $y = x^5 + C$ (b) $x^3y = x^5 + C$ so that $y = x^2 + \dfrac{C}{x^3}$.

If we consider examples of this kind in a more general setting we obtain the following Key Point:

Key Point 2

The solution of the equation

$$\frac{d}{dx}(f(x) \cdot y) = g(x)$$

is

$$f(x) \cdot y = \int g(x)\, dx \qquad \text{or} \qquad y = \frac{1}{f(x)} \int g(x)\, dx$$

4. Solving exact equations

As we have seen, the differential equation $\frac{d}{dx}(yx) = 3x^2$ has solution $y = x^2 + C/x$. In the solution, x^2 is called the **definite part** and C/x is called the **indefinite part** (containing the arbitrary constant of integration). If we take the definite part of this solution, i.e. $y_{\mathsf{d}} = x^2$, then

$$\frac{d}{dx}(y_{\mathsf{d}} \cdot x) = \frac{d}{dx}(x^2 \cdot x) = \frac{d}{dx}(x^3) = 3x^2.$$

Hence $y_{\mathsf{d}} = x^2$ is *a* solution of the differential equation.
Now if we take the indefinite part of the solution i.e. $y_{\mathsf{i}} = C/x$ then

$$\frac{d}{dx}(y_{\mathsf{i}} \cdot x) = \frac{d}{dx}\left(\frac{C}{x} \cdot x\right) = \frac{d}{dx}(C) = 0.$$

It is always the case that the general solution of an exact equation is in two parts: a definite part $y_{\mathsf{d}}(x)$ which is *a* solution of the differential equation and an indefinite part $y_{\mathsf{i}}(x)$ which satisfies a simpler version of the differential equation in which the right-hand side is zero.

Task

(a) Solve the equation

$$\frac{d}{dx}(y \cos x) = \cos x$$

(b) Verify that the indefinite part of the solution satisfies the equation

$$\frac{d}{dx}(y \cos x) = 0.$$

(a) Integrate both sides of the first differential equation:

Your solution

Answer

$$y\cos x = \int \cos x\,dx = \sin x + C \qquad \text{leading to} \qquad y = \tan x + C\sec x$$

(b) Substitute for y in the indefinite part (i.e. the part which contains the arbitrary constant) in the second differential equation:

Your solution

Answer

The indefinite part of the solution is $y_{\mathsf{i}} = C\sec x$ and so $y_{\mathsf{i}}\cos x = C$ and

$$\frac{d}{dx}(y_{\mathsf{i}}\cos x) = \frac{d}{dx}(C) = 0$$

5. Recognising an exact equation

The equation $\dfrac{d}{dx}(yx) = 3x^2$ is exact, as we have seen. If we expand the left-hand side of this equation (i.e. differentiate the product) we obtain

$$x\frac{dy}{dx} + y.$$

Hence the equation

$$x\frac{dy}{dx} + y = 3x^2$$

must be exact, but it is not so obvious that it is exact as in the original form. This leads to the following Key Point:

Key Point 3

The equation

$$f(x)\frac{dy}{dx} + y\ f'(x) = g(x)$$

is exact. It can be re-written as

$$\frac{d}{dx}(y\ f(x)) = g(x) \qquad \text{so that} \qquad y\ f(x) = \int g(x)\ dx$$

Example 4

Solve the equation

$$x^3\frac{dy}{dx} + 3x^2y = x$$

Solution

Comparing this equation with the form in Key Point 3 we see that $f(x) = x^3$ and $g(x) = x$. Hence the equation can be written

$$\frac{d}{dx}(yx^3) = x$$

which has solution

$$yx^3 = \int x\,dx = \tfrac{1}{2}x^2 + C.$$

Therefore

$$y = \frac{1}{2x} + \frac{C}{x^3}.$$

Task

Solve the equation $\sin x\,\dfrac{dy}{dx} + y\cos x = \cos x$.

Your solution

Answer

You should obtain $y = 1 + C\text{cosec}\,x$ since, here $f(x) = \sin x$ and $g(x) = \cos x$. Then

$$\frac{d}{dx}(y\sin x) = \cos x \qquad \text{and} \qquad y\sin x = \int \cos x\;dx = \sin x + C$$

Finally $y = 1 + C\,\text{cosec}\,x$.

Exercises

1. Solve the equation $\dfrac{d}{dx}(yx^2) = x^3$.
2. Solve the equation $\dfrac{d}{dx}(y\mathrm{e}^x) = \mathrm{e}^{2x}$ given the condition $y(0) = 2$.
3. Solve the equation $\mathrm{e}^{2x}\dfrac{dy}{dx} + 2\mathrm{e}^{2x}y = x^2$.
4. Show that the equation $x^2\dfrac{dy}{dx} + 2xy = x^3$ is exact and obtain its solution.
5. Show that the equation $x^2\dfrac{dy}{dx} + 3xy = x^3$ is not exact.
   Multiply the equation by x and show that the resulting equation **is** exact and obtain its solution.

Answers

1. $y = \dfrac{x^2}{4} + \dfrac{C}{x^2}$. 2. $y = \frac{1}{2}\mathrm{e}^x + \frac{3}{2}\mathrm{e}^{-x}$. 3. $y = \left(\frac{1}{3}x^3 + C\right)\mathrm{e}^{-2x}$. 4. $y = \dfrac{1}{4}x^2 + \dfrac{C}{x^2}$.
5. $y = \dfrac{1}{5}x^2 + \dfrac{C}{x^3}$.

6. The integrating factor

The equation

$$x^2\frac{dy}{dx} + 3x\ y = x^3$$

is not exact. However, if we multiply it by x we obtain the equation

$$x^3\frac{dy}{dx} + 3x^2y = x^4.$$

This can be re-written as

$$\frac{d}{dx}(x^3y) = x^4$$

which is an exact equation with solution

$$x^3y = \int x^4 dx$$

so $\qquad x^3y = \dfrac{1}{5}x^5 + C$

and hence

$$y = \frac{1}{5}x^2 + \frac{C}{x^3}.$$

The function by which we multiplied the given differential equation in order to make it exact is called an **integrating factor**. In this example the integrating factor is simply x.

Task

Which of the following differential equations can be made exact by multiplying by x^2?

(a) $\dfrac{dy}{dx} + \dfrac{2}{x}y = 4$ (b) $x\,\dfrac{dy}{dx} + 3y = x^2$ (c) $\dfrac{1}{x}\,\dfrac{dy}{dx} - \dfrac{1}{x^2}y = x$

(d) $\dfrac{1}{x}\,\dfrac{dy}{dx} + \dfrac{1}{x^2}y = 3.$

Where possible, write the exact equation in the form $\dfrac{d}{dx}(f(x)\ y) = g(x)$.

Your solution

Answer

(a) Yes. $x^2\dfrac{dy}{dx} + 2xy = 4x^2$ becomes $\dfrac{d}{dx}(x^2y) = 4x^2$.

(b) Yes. $x^3\dfrac{dy}{dx} + 3x^2y = x^4$ becomes $\dfrac{d}{dx}(x^3y) = x^4$.

(c) No. This equation is already exact as it can be written in the form $\dfrac{d}{dx}\left(\dfrac{1}{x}\ y\right) = x$.

(d) Yes. $x\dfrac{dy}{dx} + y = 3x^2$ becomes $\dfrac{d}{dx}(xy) = 3x^2$.

7. Finding the integrating factor for linear ODEs

The differential equation governing the current i in a circuit with inductance L and resistance R in series subject to a constant applied electromotive force $E\cos\omega t$, where E and ω are constants, is

$$L\,\frac{di}{dt} + Ri = E\cos\omega t \qquad (1)$$

This is an example of a linear differential equation in which i is the dependent variable and t is the independent variable. The general standard form of a linear first order differential equation is normally written with 'y' as the dependent variable and with 'x' as the independent variable and arranged so that the coefficient of $\dfrac{dy}{dx}$ is 1. That is, it takes the form:

$$\frac{dy}{dx} + f(x)\ y = g(x) \qquad (2)$$

in which $f(x)$ and $g(x)$ are functions of x.

Comparing (1) and (2), x is replaced by t and y by i to produce $\dfrac{di}{dt} + f(t)\, i = g(t)$. The function $f(t)$ is the coefficient of the dependent variable in the differential equation. We shall describe the method of finding the integrating factor for (1) and then generalise it to a linear differential equation written in standard form.

Step 1 Write the differential equation in standard form i.e. with the coefficient of the derivative equal to 1. Here we need to divide through by L:

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \cos \omega t.$$

Step 2 Integrate the coefficient of the dependent variable (that is, $f(t) = R/L$) with respect to the independent variable (that is, t), and ignoring the constant of integration

$$\int \frac{R}{L}\, dt = \frac{R}{L}\, t.$$

Step 3 Take the exponential of the function obtained in Step 2.

This is the integrating factor (I.F.)

$$\text{I.F.} = \mathrm{e}^{Rt/L}.$$

This leads to the following Key Point on integrating factors:

Key Point 4

The linear differential equation (written in standard form):

$$\frac{dy}{dx} + f(x)y = g(x) \qquad \text{has an integrating factor} \qquad \text{I.F.} = \exp\left[\int f(x)dx\right]$$

Task

Find the integrating factors for the equations

(a) $x\dfrac{dy}{dx} + 2x\, y = x\mathrm{e}^{-2x}$ (b) $t\dfrac{di}{dt} + 2t\, i = t\mathrm{e}^{-2t}$ (c) $\dfrac{dy}{dx} - (\tan x)y = 1.$

Your solution

Answer

(a) **Step 1** Divide by x to obtain $\dfrac{dy}{dx} + 2y = \mathrm{e}^{-2x}$

Step 2 The coefficient of the independent variable is 2 hence $\displaystyle\int 2\ dx = 2x$

Step 3 I.F. $= \mathrm{e}^{2x}$

(b) The only difference from (a) is that i replaces y and t replaces x. Hence I.F. $= \mathrm{e}^{2t}$.

(c) **Step 1** This is already in the standard form.

Step 2 $\displaystyle\int -\tan x\ dx = \int \frac{-\sin x}{\cos x}\ dx = \ln\cos x.$

Step 3 I.F. $= \mathrm{e}^{\ln\cos x} = \cos x$

8. Solving equations via the integrating factor

Having found the integrating factor for a linear equation we now proceed to solve the equation. Returning to the differential equation, written in standard form:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\cos\omega t$$

for which the integrating factor is

$$\mathrm{e}^{Rt/L}$$

we multiply the equation by the integrating factor to obtain

$$\mathrm{e}^{Rt/L}\ \frac{di}{dt} + \frac{R}{L}\ \mathrm{e}^{Rt/L}\ i = \frac{E}{L}\ \mathrm{e}^{Rt/L}\cos\omega t$$

At this stage the left-hand side of this equation can **always** be simplified as follows:

$$\frac{d}{dt}(\mathrm{e}^{Rt/L}\ i) = \frac{E}{L}\ \mathrm{e}^{Rt/L}\cos\omega t.$$

Now this is in the form of an exact differential equation and so we can integrate both sides to obtain the solution:

$$\mathrm{e}^{Rt/L}\ i = \frac{E}{L}\int \mathrm{e}^{Rt/L}\cos\omega t\ dt.$$

All that remains is to complete the integral on the right-hand side. Using the method of integration by parts we find

$$\int \mathrm{e}^{Rt/L}\cos\omega t\ dt = \frac{L}{L^2\omega^2 + R^2}\left[\omega L\ \sin\omega t + R\cos\omega t\right]\mathrm{e}^{Rt/L}$$

Hence

$$\mathrm{e}^{Rt/L}\ i = \frac{E}{L^2\omega^2 + R^2}\left[\omega L\sin\omega t + R\cos\omega t\right]\ \mathrm{e}^{Rt/L} + C.$$

Finally

$$i = \frac{E}{L^2\omega^2 + R^2}\left[\omega L\sin\omega t + R\cos\omega t\right] + C\ \mathrm{e}^{-Rt/L}.$$

is the solution to the original differential equation (1). Note that, as we should expect for the solution to a first order differential equation, it contains a single arbitrary constant C.

Task

Using the integrating factors found earlier in the Task on pages 22-23, find the general solutions to the differential equations
(a) $x^2\dfrac{dy}{dx}+2x^2y=x^2\mathrm{e}^{-2x}$ (b) $t^2\dfrac{di}{dt}+2t^2i=t^2\mathrm{e}^{-2t}$ (c) $\dfrac{dy}{dx}-(\tan x)y=1.$

Your solution

Answer

(a) The standard form is $\dfrac{dy}{dx}+2y=\mathrm{e}^{-2x}$ for which the integrating factor is e^{2x}.

$$\begin{aligned} \mathrm{e}^{2x}\frac{dy}{dx}+2\mathrm{e}^{2x}\,y &= 1 \\ \text{i.e.}\quad \frac{d}{dx}\left(\mathrm{e}^{2x}\,y\right) &= 1 \qquad \text{so that} \qquad \mathrm{e}^{2x}y=x+C \\ \text{leading to}\quad y &= (x+C)\mathrm{e}^{-2x} \end{aligned}$$

(b) The general solution is $i=(t+C)\mathrm{e}^{-2t}$ as this problem is the same as (a) with different variables.

(c) The equation is in standard form and the integrating factor is $\cos x$.

$$\begin{aligned} \text{then}\quad \frac{d}{dx}(\cos x\ y) &= \cos x \qquad \text{so that} \qquad \cos x\ y=\int\cos x\,dx=\sin x+C \\ \text{giving}\quad y &= \tan x+C\sec x \end{aligned}$$

Engineering Example 1

An RC circuit with a single frequency input

Introduction

The components in RC circuits containing resistance, inductance and capacitance can be chosen so that the circuit filters out certain frequencies from the input. A particular kind of filter circuit consists of a resistor and capacitor in series and acts as a high cut (or low pass) filter. The high cut frequency is defined to be the frequency at which the magnitude of the voltage across the capacitor (the output voltage) is $1/\sqrt{2}$ of the magnitude of the input voltage.

Problem in words

Calculate the high cut frequency for an RC circuit is subjected to a single frequency input of angular frequency ω and magnitude v_i.

(a) Find the steady state solution of the equation

$$R\frac{dq}{dt} + \frac{q}{C} = v_i e^{j\omega t}$$

and hence find the magnitude of

(i) the voltage across the capacitor $v_c = \dfrac{q}{C}$

(ii) the voltage across the resistor $v_R = R\,\dfrac{dq}{dt}$

(b) Using the impedance method of HELM 12.6 confirm your results to part (a) by calculating

(i) the voltage across the capacitor v_c

(ii) the voltage across the resistor v_R in response to a single frequency of angular frequency ω and magnitude v_i.

(c) For the case where $R = 1\ k\Omega$ and $C = 1\ \mu F$, find the ratio $\dfrac{|v_c|}{|v_i|}$ and complete the table below

ω	10	10^2	10^3	10^4	10^5	10^6
$\dfrac{\lvert v_c\rvert}{\lvert v_i\rvert}$						

(d) Explain why the table results show that a RC circuit acts as a high-cut filter and find the value of the high-cut frequency, defined as $f_{hc} = \omega_{hc}/2\pi$, such that $\dfrac{|v_c|}{|v_i|} = \dfrac{1}{\sqrt{2}}$.

Mathematical statement of the problem

We need to find a particular solution to the differential equation $R\,\dfrac{dq}{dt} + \dfrac{q}{C} = v_i e^{j\omega t}$.
This will give us the steady state solution for the charge q. Using this we can find $v_c = \dfrac{q}{C}$ and $v_R = R\,\dfrac{dq}{dt}$. These should give the same result as the values calculated by considering the impedances in the circuit. Finally we can calculate $\dfrac{|v_c|}{|v_i|}$ and fill in the table of values as required and find the high-cut frequency from $\dfrac{|v_c|}{|v_i|} = \dfrac{1}{\sqrt{2}}$ and $f_{hc} = \omega_{hc}/2\pi$.

Mathematical solution

(a) To find a particular solution, we try a function of the form $q = c_0 e^{j\omega t}$ which means that

$$\frac{dq}{dt} = j\omega c_0 e^{j\omega t}.$$

Substituting into $R\,\dfrac{dq}{dt} + \dfrac{q}{C} = v_i e^{j\omega t}$ we get

$$Rj\omega c_0 e^{j\omega t} + \frac{c_0 e^{j\omega t}}{C} = v_i e^{j\omega t} \quad \Rightarrow \quad Rj\omega c_0 + \frac{c_0}{C} = v_i$$

$$\Rightarrow \quad c_0 = \frac{v_i}{Rj\omega + \frac{1}{C}} = \frac{Cv_i}{RCj\omega + 1} \quad \Rightarrow \quad q = \frac{Cv_i}{RCj\omega + 1}\, e^{j\omega t}$$

Thus

(i) $v_c = \dfrac{q}{C} = \dfrac{v_i}{RCj\omega + 1}\, e^{j\omega t}$ and (ii) $v_R = \dfrac{dq}{dt} = \dfrac{RCv_i j\omega}{RCj\omega + 1}\, e^{j\omega t}$

(b) We use the impedance to determine the voltage across each of the elements. The applied voltage is a single frequency of angular frequency ω and magnitude v_i such that $V = v_i e^{j\omega t}$.

For an RC circuit, the impedance of the circuit is $Z = Z_R + Z_c$ where Z_R is the impedance of the resistor R and Z_c is the impedance of the capacitor $Z_c = -\dfrac{j}{\omega C}$.

Therefore $Z = R - \dfrac{j}{\omega C}$.

The current can be found using $v = Zi$ giving

$$v_i e^{j\omega t} = \left(R - \frac{j}{\omega C}\right) i \quad \Rightarrow \quad i = \frac{v_i e^{j\omega t}}{R - \frac{j}{\omega C}}$$

We can now use $v_c = z_c i$ and $v_R = z_R i$ giving

(i) $v_c = \dfrac{q}{C} = -\dfrac{j}{\omega C} \times \dfrac{v_i}{R - \frac{j}{\omega C}}\, e^{j\omega t} = \dfrac{v_i}{RCj\omega + 1}\, e^{j\omega t}$

(ii) $v_R = \dfrac{Rv_i}{R - \frac{j}{\omega C}}\, e^{j\omega t} = \dfrac{RCv_i j\omega}{RCj\omega + 1}\, e^{j\omega t}$

which confirms the result in part (a) found by solving the differential equation.

(c) When $R = 1000\ \Omega$ and $C = 10^{-6} F$

$$v_c = \frac{v_i}{RCj\omega + 1}\, e^{j\omega t} = \frac{v_i}{10^{-3} j\omega + 1}\, e^{j\omega t}$$

So $\dfrac{|v_c|}{|v_i|} = \left|\dfrac{1}{10^{-3}j\omega + 1}\right| \ |e^{j\omega t}| = \left|\dfrac{1}{10^{-3}j\omega + 1}\right| = \dfrac{1}{\sqrt{10^{-6}\omega^2 + 1}}$

Table 1: Values of $\left|\dfrac{v_c}{v_i}\right|$ for a range of values of ω

ω	10	10^2	10^3	10^4	10^5	10^6
$\dfrac{\lvert v_c\rvert}{\lvert v_i\rvert}$	0.99995	0.995	0.707	0.00995	0.0099995	0.001

(d) Table 1 shows that a RC circuit can be used as a high-cut filter because for low values of ω, $\dfrac{|v_c|}{|v_i|}$ is approximately 1 and for high values of ω, $\dfrac{|v_c|}{|v_i|}$ is approximately 0. So the circuit will filter out high frequency values.

$$\frac{|v_c|}{|v_i|} = \frac{1}{\sqrt{2}} \text{ when } \frac{1}{\sqrt{10^{-6}\omega^2 + 1}} = \frac{1}{\sqrt{2}} \quad \Leftrightarrow \quad 10^{-6}\omega^2 + 1 = 2 \Leftrightarrow 10^{-6}\omega^2 = 1 \Leftrightarrow \omega^2 = 10^6$$

As we are considering ω to be a positive frequency, $\omega = 1000.$

So $f_{hc} = \dfrac{\omega_{hc}}{2\pi} = \dfrac{1000}{2\pi} \approx 159$ Hz.

Interpretation

We have shown that for an RC circuit finding the steady state solution of the differential equation with a single frequency input voltage yields the same result for $\dfrac{|v_c|}{|v_i|}$ and $\dfrac{|v_R|}{|v_i|}$ as found by working with the complex impedances for the circuit.

An RC circuit can be used as a high-cut filter and in the case where $R = 1\ k\Omega, C = 1\ \mu F$ we found the high-cut frequency to be at approximately 159 Hz.

This means that the circuit will pass frequencies less than this value and remove frequencies greater than this value.

Exercises

1. Solve the equation $x^2\dfrac{dy}{dx} + x\ y = 1$.
2. Find the solution of the equation $x\dfrac{dy}{dx} - y = x$ subject to the condition $y(1) = 2$.
3. Find the general solution of the equation $\dfrac{dy}{dt} + (\tan t)\ y = \cos t$.
4. Solve the equation $\dfrac{dy}{dt} + (\cot t)\ y = \sin t$.
5. The temperature θ (measured in degrees) of a body immersed in an atmosphere of varying temperature is given by $\dfrac{d\theta}{dt} + 0.1\theta = 5 - 2.5t$. Find the temperature at time t if $\theta = 60°$C when $t = 0$.
6. In an LR circuit with applied voltage $E = 10(1 - \mathrm{e}^{-0.1t})$ the current i is given by

   $$L\frac{di}{dt} + Ri = 10(1 - \mathrm{e}^{-0.1t}).$$

   If the initial current is i_0 find i subsequently.

Answers

1. $y = \dfrac{1}{x}\ln x + \dfrac{C}{x}$
2. $y = x\ln x + 2x$
3. $y = (t + C)\cos t$
4. $y = \left(\frac{1}{2}\ t - \frac{1}{4}\sin 2t + C\right)\mathrm{cosec}\, t$
5. $\theta = 300 - 25t - 240\mathrm{e}^{-0.1t}$
6. $i = \dfrac{10}{R} - \left(\dfrac{100}{10R - L}\right)\mathrm{e}^{-0.1t} + \left[i_0 + \dfrac{10L}{R(10R - L)}\right]\mathrm{e}^{-Rt/L}$