Contents 27

Multiple Integration

Learning outcomes

In this Workbook you will learn to integrate a function of two variables over various rectangular and non-rectangular areas. You will learn how to do this for various other coordinate systems. You will learn to integrate a function of three variables over a volume.

Introduction to Surface Integrals

27.1

Introduction

Often in Engineering it is necessary to find the sum of a quantity over an area or surface. This can be achieved by means of a surface integral also known as a double integral i.e. a function is integrated twice, once with respect to one variable and subsequently with respect to another variable. This Section looks at the concept of the double integral and how to evaluate a double integral over a rectangular area.

Prerequisites

Before starting this Section you should . . .

- thoroughly understand the various techniques of integration
- be familiar with the concept of a function of two variables

Learning Outcomes

On completion you should be able to . . .

- understand the concept of a surface integral
- integrate a function over a rectangular region

1. An example of a surface integral

An engineer involved with the construction of a dam to hold back the water in a reservoir needs to be able to calculate the total force the water exerts on the dam so that the dam is built with sufficient strength.

In order to calculate this force, two results are required:

    (a) The pressure p of the water is proportional to the depth. That is

$$p = kd \tag{1}$$

       where k is a constant.

    (b) The force on an area subjected to constant pressure is given by

$$\text{force} = \text{pressure} \times \text{area} \tag{2}$$

The diagram shows the face of the dam. The depth of water is h and δA is a small area in the face of the dam with coordinates (x, y).
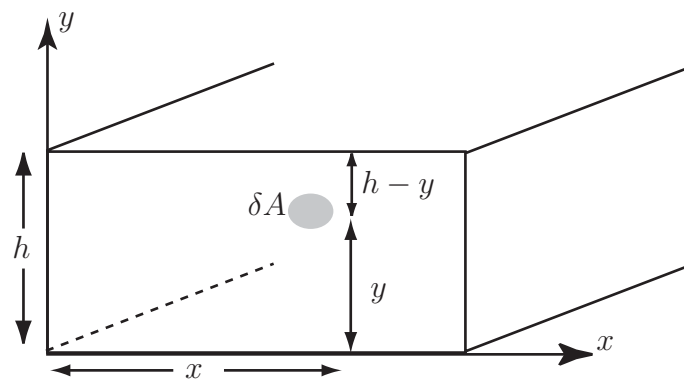

Figure 1

Using (1), the pressure at $\delta A \sim k(h - y)$. Using (2), the force on an area $\delta A \sim k(h - y)\delta A$.
Both of these expressions are approximate as y is slightly different at the top of δA to the bottom.
Now

$$\text{Total force on dam} \quad = \quad \text{sum of forces on all areas} \ \ \delta A \text{ making up the face of the dam}$$

$$\approx \ \sum_{\text{all } \delta A} k(h - y)\delta A$$

For a better approximation let δA become smaller, and for the exact result find the limit as $\delta A \to 0$. Then

$$\text{Total force on the dam} \quad = \quad \lim_{\delta A \to 0} \sum k(h - y)\delta A$$

$$= \quad \int_A k(h - y) \ dA$$

where $\displaystyle\int_A k(h - y) \ dA$ stands for the **surface integral** of $k(h - y)$ over the area A. Surface integrals are evaluated using **double integrals**. The following Section shows a double integral being developed in the case of the volume under a surface.

2. Single and double integrals

As has been seen in HELM 14.3, the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is given by $\displaystyle\int_a^b f(x)\,dx$ (assuming that the curve lies above the axis for all x in the range $a \leq x \leq b$). This is illustrated by the figure below.
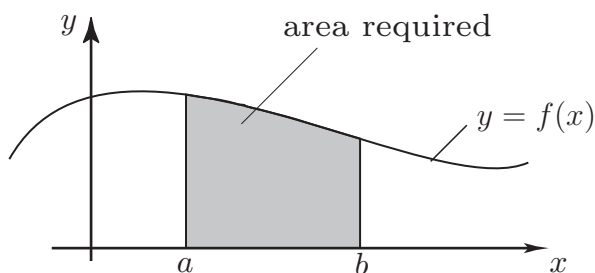
Figure 2

In a similar manner, the volume under a surface (given by a function of two variables $z = f(x, y)$) and above the xy plane can be found by integrating the function $z = f(x, y)$ twice, once with respect to x and once with respect to y.
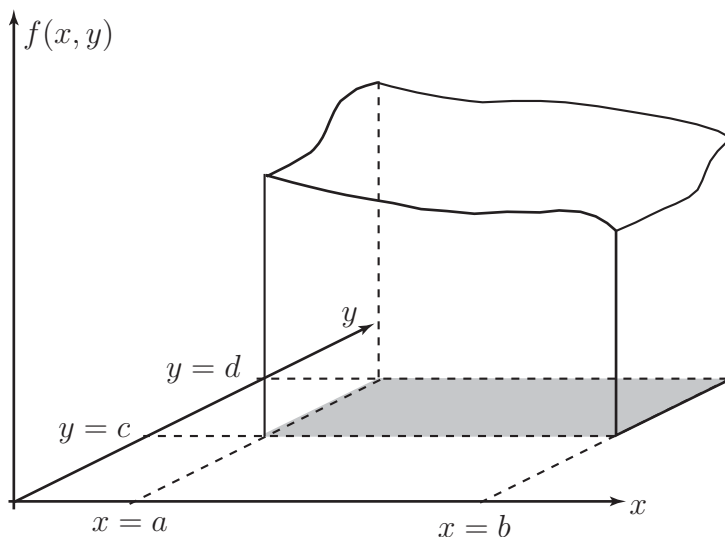
Figure 3

The above figure shows the part of a surface given by $f(x, y)$ which lies above the rectangle $a \leq x \leq b$, $c \leq y \leq d$. This rectangle is shaded and the volume above this rectangle but below the surface can be seen.
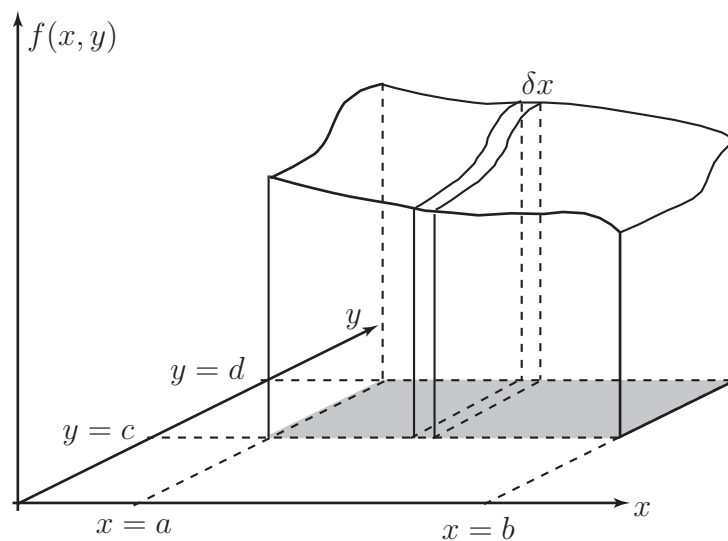
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Figure 4

Imagine a vertical slice taken through this volume at right angles to the x-axis (figure above). This slice has thickness δx and lies at position x. Assuming that δx is small enough that the areas of both sides (left and right) of this slice are virtually the same, the area of each face of the slice is given by the integral

$$\int_{y=c}^{y=d} f(x,y) \ dy \qquad \text{(where } x \text{ measures the position of the slice)}$$

and the volume of the slice will be given by

$$\delta x \int_{y=c}^{y=d} f(x,y) \ dy$$

To find the total volume between the surface and the xy plane, this quantity should be summed over all possible such slices, each for a different value of x. Thus

$$V \approx \sum_{i} \int_{y=c}^{y=d} f(x_i, y) \ dy \ \delta x$$

When δx becomes infinitesmally small, it can be considered to be dx and the summation will change into an integral. Hence

$$V = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y) \ dy dx$$

Thus the volume is given by integrating the function twice, once with respect to x and once with respect to y.

The procedure shown here considers the volume above a rectangular area and below the surface. The volume beneath the surface over a non-rectangular area can also be found by integrating twice (see Section 27.2).

> ### Key Point 1
>
> #### Volume Integral
>
> The volume under the surface $z = f(x, y)$ and above a rectangular region in the xy plane (that is the rectangle $a \leq x \leq b, \quad c \leq y \leq d$) is given by the integral:
>
> $$V = \int_{x=a}^{b} \int_{y=c}^{d} f(x, y) \; dy dx$$

3. 'Inner' and 'Outer' integrals

A typical double integral may be expressed as

$$I = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x, y) \; dy \right] \; dx$$

where the part in the centre i.e.

$$\int_{y=c}^{y=d} f(x, y) \; dy$$

(known as the inner integral) is the integral of a function of x and y with respect to y. As the integration takes place with respect to y, the variable x may be regarded as a fixed quantity (a constant) but for every different value of x, the inner integral will take a different value. Thus, the inner integral will be a function of x e.g. $g(x) = \int_{y=c}^{y=d} f(x, y) \; dy$.

This inner integral, being a function of x, once evaluated, can take its place within the outer integral i.e. $I = \int_{x=a}^{x=b} g(x) \; dx$ which can then be integrated with respect to x to give the value of the double integral.

The limits on the outer integral will be constants; the limits on the inner integral may be constants (in which case the integration takes place over a rectangular area) or may be functions of the variable used for the outer integral (in this case x). In this latter case, the integration takes place over a non-rectangular area (see Section 27.2). In the Examples quoted in this Section or in the early parts of the next Section, the limits include the name of the relevant variable; this can be omitted once more familiarity has been gained with the concept. It will be assumed that the limits on the inner integral apply to the variable used to integrate the inner integral and the limits on the outer integral apply to the variable used to integrate this outer integral.

4. Integration over rectangular areas

Consider the double integral

$$I = \int_{x=0}^{5} \int_{y=-1}^{1} (2x + y) \; dy\,dx$$

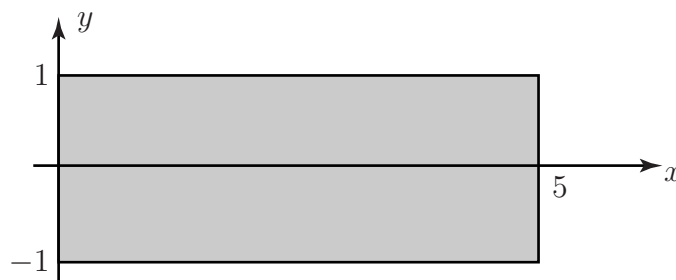This represents an integral over the rectangle shown below.

Figure 5

Here, the inner integral is

$$g(x) = \int_{-1}^{1} (2x + y) \; dy$$

and the outer integral is

$$I = \int_{x=0}^{5} g(x) \; dx$$

Looking in more detail at the inner integral

$$g(x) = \int_{-1}^{1} (2x + y) \; dy$$

the function $(2x + y)$ can be integrated with respect to y (keeping x constant) to give $2xy + \frac{1}{2}y^2 + C$ (where C is a constant and can be omitted as the integral is a definite integral) i.e.

$$g(x) = \left[2xy + \frac{1}{2}y^2 \right]_{-1}^{1} = \left(2x + \frac{1}{2} \right) - \left(-2x + \frac{1}{2} \right) = 2x + \frac{1}{2} + 2x - \frac{1}{2} = 4x.$$

This is a function of x as expected. This inner integral can be placed into the outer integral to get

$$I = \int_{x=0}^{5} 4x \; dx$$

which becomes

$$I = \left[2x^2 \right]_{0}^{5} = 2 \times 5^2 - 2 \times 0^2 = 2 \times 25 - 0 = 50$$

Hence the double integral

$$I = \int_{x=0}^{5} \int_{y=-1}^{1} (2x + y) \; dy\,dx = 50$$

Key Point 2

Double Integral

When evaluating a double integral, evaluate the inner integral first and substitute the result into the outer integral.

Example 1

Evaluate the double integral $$I = \int_{x=-1}^{2} \int_{y=-2}^{3} x^2 y \; dy dx$$

This integral is evaluated over the area shown below.

Figure 6

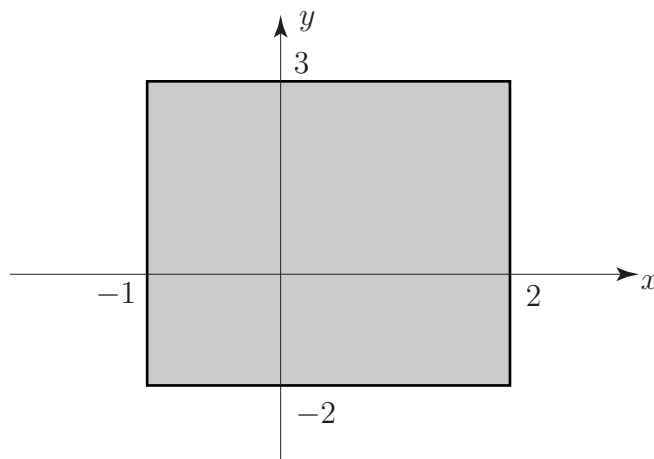**Solution**

Here, the inner integral is

$$g(x) = \int_{y=-2}^{3} x^2 y \; dy = \left[x^2 \frac{y^2}{2} \right]_{-2}^{3} = \frac{9}{2} x^2 - \frac{4}{2} x^2 = \frac{5}{2} x^2$$

and hence the outer integral is

$$I = \int_{x=-1}^{2} \frac{5}{2} x^2 \; dx = \left[\frac{5}{2} \frac{1}{3} x^3 \right]_{-1}^{2} = \frac{5}{6} \times 8 - \frac{5}{6} (-1) = \frac{15}{2}$$

Example 2

Use the above approach to evaluate the double integral

$$I = \int_{x=0}^{5} \int_{y=-1}^{1} x^2 \cos \frac{\pi y}{2} \ dy dx$$

Note that the limits are the same as in a previous case but that the function itself has changed.

Solution

The inner integral is

$$\int_{y=-1}^{1} x^2 \cos \frac{\pi y}{2} \ dy = \left[\frac{2}{\pi} x^2 \sin \frac{\pi y}{2} \right]_{-1}^{1} = \frac{2}{\pi} x^2 1 - \frac{2}{\pi} x^2 (-1) = \frac{4}{\pi} x^2$$

so the outer integral becomes

$$I = \int_{x=0}^{5} \frac{4}{\pi} x^2 \ dx = \left[\frac{4}{3\pi} x^3 \right]_{0}^{5} = \frac{4}{3\pi} 125 - \frac{4}{3\pi} 0 = \frac{500}{3\pi} \approx 53.1$$

Clearly, variables other than x and y may be used.

Example 3

Evaluate the double integral

$$I = \int_{s=1}^{4} \int_{t=0}^{\pi} s \sin t \ dt ds$$

Solution

This integral becomes (dispensing with the step of formally writing the inner integral),

$$\begin{aligned} I &= \int_{s=1}^{4} \left[-s \cos t \right]_{0}^{\pi} ds = \int_{1}^{4} \left[-s \cos \pi + s \cos 0 \right] ds = \int_{1}^{4} \left[-s (-1) + s (1) \right] ds \\ &= \int_{1}^{4} 2s \ ds = \left[s^2 \right]_{1}^{4} = 16 - 1 = 15 \end{aligned}$$

Clearly, evaluating the integrals can involve further tools of integration, e.g. integration by parts or by substitution.

Example 4

Evaluate the double integral

$$I = \int_{-1}^{2} \int_{-2}^{3} \frac{xye^{-x}}{y^2 + 1} \ dydx$$

Here, the limits have not formally been linked with a variable name but the limits on the outer integral apply to x and the limits on the inner integral apply to y. As the integrations are more complicated, the inner integral will be evaluated explicitly.

Solution

$$\text{Inner integral} = \int_{-2}^{3} \frac{xye^{-x}}{y^2 + 1} \ dy$$

which can be evaluated by means of the substitution $U = y^2 + 1$.

If $U = y^2 + 1$ then $dU = 2y \ dy$ so $y \ dy = \frac{1}{2}dU$.

Also if $y = -2$ then $U = 5$ and if $y = 3$ then $U = 10$.
So the inner integral becomes (remembering that x may be treated as a constant)

$$\int_{5}^{10} \frac{1}{2} \frac{xe^{-x}}{U} \ dU = \frac{xe^{-x}}{2} \int_{5}^{10} \frac{dU}{U} = \frac{xe^{-x}}{2} \left[\ln U \right]_{5}^{10} = \frac{xe^{-x}}{2} \left(\ln 10 - \ln 5 \right) = xe^{-x} \frac{\ln 2}{2}$$

and so the double integral becomes

$$I = \int_{-1}^{2} xe^{-x} \frac{\ln 2}{2} \ dx = \frac{\ln 2}{2} \int_{-1}^{2} xe^{-x} \ dx$$

which can be evaluated by integration by parts.

$$\begin{aligned} I &= \frac{\ln 2}{2} \left[\left[-xe^{-x} \right]_{-1}^{2} - \int_{-1}^{2} 1 \times \left(-e^{-x} \right) \ dx \right] = \frac{\ln 2}{2} \left[-2e^{-2} + (-1) \, e^{1} + \int_{-1}^{2} e^{-x} \ dx \right] \\ &= \frac{\ln 2}{2} \left[-2e^{-2} - e^{1} + \left[-e^{-x} \right]_{-1}^{2} \right] \\ &= \frac{\ln 2}{2} \left[-2e^{-2} - e^{-1} - e^{-2} + e^{1} \right] = \frac{\ln 2}{2} \left[-3e^{-2} \right] \approx -0.14 \end{aligned}$$

Task

Evaluate the following double integral.

$$I = \int_{-1}^{1} \int_{0}^{2} \left(x^2 y + 3y^2 \right) \ dy dx$$

Your solution

Answer

The inner integral $= \int_{0}^{2} \left(x^2 y + 3y^2 \right) dy = \left[\frac{1}{2} x^2 y^2 + y^3 \right]_{0}^{2} = \frac{1}{2} \times 4x^2 + 8 - (0 + 0) = 2x^2 + 8$

This can be put in the outer integral to give

$$I = \int_{-1}^{1} \left(2x^2 + 8 \right) \ dx = \left[\frac{2}{3} x^3 + 8x \right]_{-1}^{1} = \frac{2}{3} + 8 - \left(-\frac{2}{3} - 8 \right) = \frac{4}{3} + 16 = \frac{52}{3}$$

Exercises

Evaluate the following double integrals over rectangular areas.

1. $I = \int_{x=0}^{1} \int_{y=0}^{2} xy \ dy dx$

2. $I = \int_{-2}^{3} \int_{0}^{4} \left(x^2 + y^2 \right) \ dx dy$

3. $I = \int_{0}^{\pi} \int_{-1}^{1} y \sin^2 x \ dy dx$

4. $I = \int_{0}^{2} \int_{-1}^{3} st^3 \ ds dt$

5. $I = \int_{0}^{3} \int_{0}^{1} 5z^2 w \left(w^2 - 1 \right)^4 \ dw dz$ (Requires integration by substitution.)

6. $I = \int_{0}^{2\pi} \int_{0}^{1} ty \sin t \ dy dt$ (Requires integration by parts.)

Answers

1. 1,   2. 460/3,   3. 0,   4. 16,   5. 9/2,   6. $-\pi$

5. Special cases

If the integrand can be written as

$$f(x, y) = g(x) h(y)$$

then the double integral

$$I = \int_a^b \int_c^d g(x) h(y) \; dy dx$$

can be written as

$$I = \int_a^b g(x) \; dx \times \int_c^d h(y) \; dy$$

i.e. the **product** of the two individual integrals. For example, the integral

$$I = \int_{x=-1}^2 \int_{y=-2}^3 x^2 y \; dy dx$$

which was evaluated earlier can be written as

$$I = \int_{x=-1}^2 x^2 \; dx \times \int_{y=-2}^3 y \; dy = \left[\frac{x^3}{3} \right]_{-1}^2 \left[\frac{y^2}{2} \right]_{-2}^3 = \left[\frac{8}{3} - \frac{(-1)}{3} \right] \left[\frac{9}{2} - \frac{4}{2} \right]$$

$$= 3 \times \frac{5}{2} = \frac{15}{2}$$

the same result as before.

Key Point 3

Double Integral as a Product

The integral

$$\int_a^b \int_c^d g(x) h(y) \; dy dx \quad \text{can be written as} \quad \int_a^b g(x) \; dx \quad \times \quad \int_c^d h(y) \; dy$$

Imagine the integral

$$I = \int_{-1}^1 \int_0^1 x e^{-y^2} \; dy dx$$

Approached directly, this would involve evaluating the integral $\int_0^1 x e^{-y^2} \; dy$ which cannot be done by algebraic means (i.e. it can only be determined numerically).

However, the integral can be re-written as

$$I = \int_{-1}^{1} x \, dx \times \int_{0}^{1} e^{-y^2} \, dy = \left[\frac{1}{2}x^2\right]_{-1}^{1} \times \int_{0}^{1} e^{-y^2} \, dy = 0 \times \int_{0}^{1} e^{-y^2} \, dy = 0$$

and the result can be found without the need to evaluate the difficult integral.

If the integrand is independent of one of the variables and is simply a function of the other variable, then only one integration need be carried out.

The integral $I_1 = \int_{a}^{b} \int_{c}^{d} h(y) \, dydx$ may be written as $I_1 = (b - a) \int_{c}^{d} h(y) \, dy$ and the integral

$I_2 = \int_{a}^{b} \int_{c}^{d} g(x) \, dydx$ may be written as $I_2 = (d - c) \int_{a}^{b} g(x) \, dx$ i.e. the integral in the variable upon which the integrand depends multiplied by the length of the range of integration for the other variable.

Example 5

Evaluate the double integral

$$I = \int_{0}^{2} \int_{-1}^{2} y^2 \, dydx$$

Solution

As the integral in y can be multiplied by the range of integration in x, the double integral will equal

$$I = (2 - 0) \int_{-1}^{2} y^2 \, dy = 2 \left[\frac{y^3}{3}\right]_{-1}^{2} = 2 \left[\frac{2^3}{3} - \frac{(-1)^3}{3}\right] = 6$$

Note that the two integrations can be carried out in either order as long as the limits are associated with the correct variable. For example

$$I = \int_{x=0}^{1} \int_{y=-1}^{2} x^4 y \, dydx = \int_{x=0}^{1} \left[\frac{x^4 y^2}{2}\right]_{-1}^{2} dx = \int_{x=0}^{1} \left[2x^4 - \frac{1}{2}x^4\right] dx$$

$$= \int_{0}^{1} \frac{3}{2} x^4 \, dx = \left[\frac{3}{10}x^5\right]_{0}^{1} = \frac{3}{10} \times 1 - \frac{3}{10} \times 0 = \frac{3}{10}$$

and

$$I = \int_{y=-1}^{2} \int_{x=0}^{1} x^4 y \, dxdy = \int_{y=-1}^{2} \left[\frac{x^5 y}{5}\right]_{0}^{1} dy = \int_{-1}^{2} \left[\frac{y}{5} - 0\right] dy$$

$$= \int_{-1}^{2} \frac{y}{5} \, dy = \left[\frac{y^2}{10}\right]_{-1}^{2} = \frac{4}{10} - \frac{1}{10} = \frac{3}{10}$$

Task

Evaluate the following integral:

$$I = \int_0^1 \int_{-1}^1 z\,(w+1)\ dwdz.$$

Your solution

Answer

1

Exercises

1. Evaluate the following integrals:

   (a) $I = \int_0^{\pi/2} \int_0^1 (y\cos x)\ dydx$

   (b) $I = \int_{-8}^3 \int_{-1}^1 y^2\ dydx$

   (c) $I = \int_0^1 \int_0^5 (s+1)^4\ dtds$

2. Evaluate the integrals $\int_{-1}^3 \int_0^2 x^3 y\ dydx$ and $\int_0^2 \int_{-1}^3 x^3 y\ dxdy$ and show that they are equal.
   As explained in the text, the order in which these integrations are carried out does not matter for integrations over rectangular areas.

Answers

1. (a) 1/2,   (b) 22/3,   (c) 31

2. 40

6. Applications of surface integration over rectangular areas

Force on a dam

At the beginning of this Section, the total force on a dam was given by the surface integral

$$\int_A k(h - y) \ dA$$

Imagine that the dam is rectangular in profile with a width of 100 m and a height h of 40 m. The expression dA is replaced by $dxdy$ and the limits on the variables x and y are 0 to 100 m and 0 to 40 m respectively. The constant k may be assumed to be 10^4 kg m^{-2} s^{-2}. The surface integral becomes the double integral

$$\int_0^{40} \int_0^{100} k(h - y) \ dxdy \qquad \text{that is} \qquad \int_0^{40} \int_0^{100} 10^4(40 - y) \ dxdy$$

As the integral in this double integral does not contain x, the integral may be written

$$
\begin{aligned}
\int_0^{40} \int_0^{100} 10^4(40 - y) \ dxdy &= (100 - 0)\int_0^{40} 10^4(40 - y) \ dy \\
&= 100 \times 10^4 \left[40y - \frac{y^2}{2} \right]_0^{40} \\
&= 10^6[(40 \times 40 - 40^2/2) - 0] \\
&= 10^6 \times 800 = 8 \times 10^8 \text{ N}
\end{aligned}
$$

that is the total force is 800 meganewtons.

Centre of pressure

We wish to find the centre of pressure (x_p, y_p) of a plane area immersed vertically in a fluid. Take the x axis to be in the surface of the fluid and the y axis to be vertically down, so that the plane Oxy contains the area.
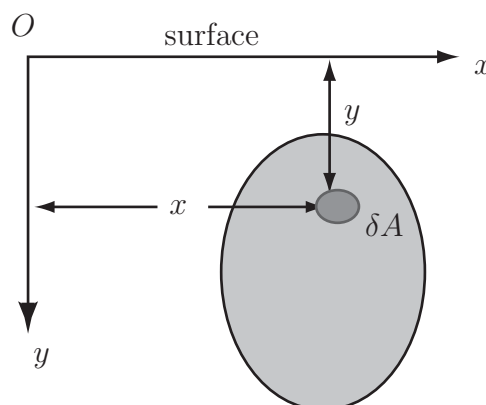

Figure 7

We require the following results:

(a) The pressure p is proportional to the depth h, so that $p = \omega h$ where ω is a constant.

(b) The force F on an area δA subjected to constant pressure p is given by $F = p\delta A$

Consider a small element of area δA at the position shown. The pressure at δA is ωy. Then the force acting on δA is $\omega y \delta A$. Hence the total force acting on the area A is $\displaystyle\int_A \omega y \ dA = \omega \int_A y \ dA$.

$$
\begin{aligned}
\text{Moment of force on } \delta A \text{ about } Oy &= \omega x y \delta A \\
\text{Total moment of force on } \delta A \text{ about } Oy &= \omega \int_A xy \ dA \\
\text{Moment of force on } \delta A \text{ about } Ox &= \omega y^2 \delta A \\
\text{Total moment of force on } \delta A \text{ about } Ox &= \omega \int_A y^2 \ dA
\end{aligned}
$$

Taking moments about Oy:

$$
\begin{aligned}
\text{total force} \times x_p &= \text{total moment} \\
\left(\omega \int_A y \ dA \right) x_p &= \omega \int_A xy \ dA \\
x_p \int_A y \ dA &= \int_A xy \ dA
\end{aligned}
$$

Taking moments about Ox:

$$
\begin{aligned}
\text{total force} \times y_p &= \text{total moment} \\
\left(\omega \int_A y \ dA \right) y_p &= \omega \int_A y^2 \ dA \\
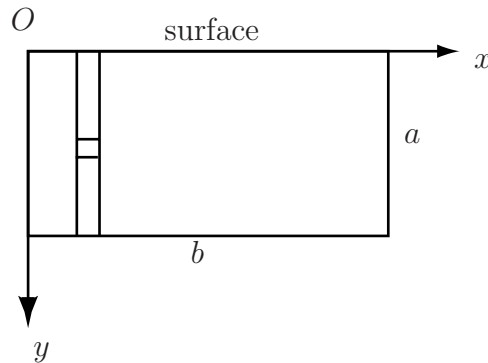y_p \int_A y \ dA &= \int_A y^2 \ dA
\end{aligned}
$$

Hence

$$
x_p = \frac{\displaystyle\int_A xy \ dA}{\displaystyle\int_A y \ dA} \quad \text{and} \quad y_p = \frac{\displaystyle\int_A y^2 \ dA}{\displaystyle\int_A y \ dA}.
$$

Example 6

A rectangle of sides a and b is immersed vertically in a fluid with one of its edges in the surface as shown in Figure 8. Where is the centre of pressure?

Figure 8

Solution

To express the surface integral as double integrals we will use cartesian coordinates and vertical slices. We need the following integrals.

$$
\int_A y \ dA = \int_0^b \int_0^a y \ dy dx = \int_0^b \left[\frac{1}{2}y^2\right]_0^a \ dx = \int_0^b \frac{1}{2}a^2 \ dx = \left[\frac{1}{2}a^2 x\right]_0^b = \frac{1}{2}a^2 b
$$

$$
\int_A xy \ dA = \int_0^b \int_0^a xy \ dy dx = \int_0^b \left[\frac{1}{2}xy^2\right]_0^a \ dx = \int_0^b \frac{1}{2}xa^2 \ dx = \left[\frac{1}{4}x^2 a^2\right]_0^b = \frac{1}{4}a^2 b^2
$$

$$
\int_A y^2 \ dA = \int_0^b \int_0^a y^2 \ dy dx = \int_0^b \left[\frac{1}{3}y^3\right]_0^a \ dx = \int_0^b \frac{1}{3}a^3 \ dx = \left[\frac{1}{3}a^3 x\right]_0^b = \frac{1}{3}a^3 b
$$

Hence

$$
y_p = \frac{\displaystyle\int_A y^2 \ dA}{\displaystyle\int_A y \ dA} = \frac{\frac{1}{3}a^3 b}{\frac{1}{2}a^2 b} = \frac{2}{3}a \quad \text{and} \quad x_p = \frac{\displaystyle\int_A xy \ dA}{\displaystyle\int_A y \ dA} = \frac{\frac{1}{4}a^2 b^2}{\frac{1}{2}a^2 b} = \frac{1}{2}b
$$

The centre of pressure is $(\frac{1}{2}b, \frac{2}{3}a)$, so is at a depth of $\frac{2}{3}a$.

Areas and moments

The surface integral $\int_A f(x, y) \; dA$ can represent a number of physical quantities, depending on the function $f(x, y)$ that is used.

Properties:

(a) If $f(x, y) = 1$ then the integral represents the area of A.

(b) If $f(x, y) = x$ then the integral represents the first moment of A about the y axis.

(c) If $f(x, y) = y$ then the integral represents the first moment of A about the x axis.

(d) If $f(x, y) = x^2$ then the integral represents the second moment of A about the y axis.

(e) If $f(x, y) = y^2$ then the integral represents the second moment of A about the x axis.

(f) If $f(x, y) = x^2 + y^2$ then the integral represents the second moment of A about the z axis.

Example 7

Given a rectangular lamina of length ℓ, width b, thickness t (small) and density ρ (see Figure 9), find the second moment of area of this lamina (moment of inertia) about the x-axis.
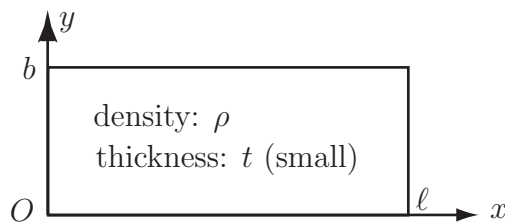

Figure 9

Solution

By property (e) above, the moment of inertia is given by

$$
\int_0^b \int_0^\ell y^2 \rho t \; dx dy = \rho t (\ell - 0) \int_0^b y^2 \; dy
$$

$$
= \ell \rho t \left[\frac{y^3}{3} \right]_0^b
$$

$$
= \ell \rho t \frac{b^3}{3}
$$

As the mass of the lamina is $M = \ell b t \rho$, the moment of inertia simplifies to $\frac{1}{3} M b^2$. The t and ρ are included in the integral to make it a moment of inertia rather than simply a second moment.

Task

By a similar method to that in Example 7, find the moment of inertia of the same lamina about the y-axis.

Your solution

Answer

From property (d) above, the moment of inertia (or second moment of area) is given by the integral

$$
\begin{aligned}
\int_0^l \int_0^b x^2 \rho t \; dydx &= \rho t(b-0)\int_0^l x^2 \; dx \\
&= b\rho t \left[\frac{x^3}{3}\right]_0^l \\
&= b\rho t \frac{l^3}{3}
\end{aligned}
$$

As the mass of the lamina is $M = lb\rho t$, the moment of inertia simplifies to $\frac{1}{3}Ml^2$. Again, the t and ρ are included in the integral to make it a moment of inertia rather than simple a second moment.

Exercises

By making use of the form of the integrand, evaluate the following double integrals:

1. $I = \displaystyle\int_0^\pi \int_0^1 y\cos^2 x \quad dydx$

2. $I = \displaystyle\int_{-8}^3 \int_{-1}^1 y^2 \; dydx$

3. $I = \displaystyle\int_0^1 \int_0^5 (s+1)^4 \, dtds$

Answers 1. $\frac{\pi}{4}$, 2. $\frac{22}{3}$ 3. 31