Integration using trig identities or a trig substitution

Some integrals involving trigonometric functions can be evaluated by using the trigonometric identities. These allow the integrand to be written in an alternative form which may be more amenable to integration.

On occasions a trigonometric substitution will enable an integral to be evaluated.

Both of these topics are described in this unit.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- use trigonometric identities to integrate $\sin^2 x$, $\cos^2 x$, and functions of the form $\sin 3x\ \cos 4x$.
- integrate products of sines and cosines using a mixture of trigonometric identities and integration by substitution
- use trigonometric substitutions to evaluate integrals

Contents

1. Introduction

By now you should be well aware of the important results that

$$\int \cos kx \, \mathrm{d}x = \frac{1}{k} \sin kx + c \qquad \int \sin kx \, \mathrm{d}x = -\frac{1}{k} \cos kx + c$$

However, a little more care is needed when we wish to integrate more complicated trigonometric functions such as $\int \sin^2 x \, \mathrm{d}x$, $\int \sin 3x \cos 2x \, \mathrm{d}x$, and so on. In case like these trigonometric identities can be used to write the integrand in an alternative form which can be integrated more readily.

Sometimes, use of a trigonometric substitution enables an integral to be found. Such substitutions are described in Section 4.

2. Integrals requiring the use of trigonometric identities

The trigonometric identities we shall use in this section, or which are required to complete the Exercises, are summarised here:

$$\begin{aligned}
2 \sin A \cos B &= \sin(A+B) + \sin(A-B) \\
2 \cos A \cos B &= \cos(A-B) + \cos(A+B) \\
2 \sin A \sin B &= \cos(A-B) - \cos(A+B) \\
\sin^2 A + \cos^2 A &= 1 \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A \\
\sin 2A &= 2 \sin A \cos A \\
1 + \tan^2 A &= \sec^2 A
\end{aligned}$$

Some commonly needed trigonometric identities

Example

Suppose we wish to find $\int_0^\pi \sin^2 x \, \mathrm{d}x$.

The strategy is to use a trigonometric identity to rewrite the integrand in an alternative form which does not include powers of $\sin x$. The trigonometric identity we shall use here is one of the 'double angle' formulae:

$$\cos 2A = 1 - 2\sin^2 A$$

By rearranging this we can write

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

Notice that by using this identity we can convert an expression involving $\sin^2 A$ into one which has no powers in. Therefore, our integral can be written

$$\int_0^\pi \sin^2 x \, \mathrm{d}x = \int_0^\pi \frac{1}{2}(1 - \cos 2x) \mathrm{d}x$$

and this can be evaluated as follows:

$$\begin{aligned}\int_0^\pi \frac{1}{2}(1-\cos 2x)\mathrm{d}x &= \left[\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)\right]_0^\pi \\ &= \left[\frac{1}{2}x-\frac{1}{4}\sin 2x\right]_0^\pi \\ &= \frac{\pi}{2}\end{aligned}$$

Example

Suppose we wish to find $\displaystyle\int \sin 3x \cos 2x \,\mathrm{d}x$.

Note that the integrand is a product of the functions $\sin 3x$ and $\cos 2x$. We can use the identity $2\sin A\cos B = \sin(A+B)+\sin(A-B)$ to express the integrand as the sum of two sine functions. With $A = 3x$ and $B = 2x$ we have

$$\begin{aligned}\int \sin 3x\cos 2x \,\mathrm{d}x &= \frac{1}{2}\int(\sin 5x+\sin x)\mathrm{d}x \\ &= \frac{1}{2}\left(-\frac{1}{5}\cos 5x-\cos x\right)+c \\ &= -\frac{1}{10}\cos 5x-\frac{1}{2}\cos x+c\end{aligned}$$

Exercises 1

Use the trigonometric identities stated on page 2 to find the following integrals.

1. (a) $\displaystyle\int \cos^2 x \,\mathrm{d}x$ (b) $\displaystyle\int_0^{\pi/2} \cos^2 x \,\mathrm{d}x$ (c) $\displaystyle\int \sin 2x\cos 2x \,\mathrm{d}x$

2. (a) $\displaystyle\int_{\pi/6}^{\pi/3} 2\cos 5x\,\cos 3x \,\mathrm{d}x$ (b) $\displaystyle\int (\sin^2 t+\cos^2 t)\mathrm{d}t$ (c) $\displaystyle\int \sin 7t\,\sin 4t \,\mathrm{d}t$.

3. Integrals involving products of sines and cosines

In this section we look at integrals of the form $\displaystyle\int \sin^m x\,\cos^n x \,\mathrm{d}x$. In the first example we see how to deal with integrals in which m is odd.

Example

Suppose we wish to find $\displaystyle\int \sin^3 x\cos^2 x \,\mathrm{d}x$.

Study of the integrand, and the table of identities shows that there is no obvious identity which will help us here. However what we will do is rewrite the term $\sin^3 x$ as $\sin x\sin^2 x$, and use the identity $\sin^2 x = 1-\cos^2 x$. The reason for doing this will become apparent.

$$\begin{aligned}\int \sin^3 x\cos^2 x \,\mathrm{d}x &= \int(\sin x\cdot\sin^2 x)\cos^2 x \,\mathrm{d}x \\ &= \int \sin x(1-\cos^2 x)\cos^2 x \,\mathrm{d}x\end{aligned}$$

At this stage the substitution $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$ enables us to rapidly complete the solution:
We find

$$\begin{aligned}\int \sin x(1-\cos^2 x)\cos^2 x\,\mathrm{d}x &= -\int (1-u^2)u^2\,\mathrm{d}u \\ &= \int (u^4-u^2)\mathrm{d}u \\ &= \frac{u^5}{5}-\frac{u^3}{3}+c \\ &= \frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+c\end{aligned}$$

In the case when m is even and n is odd we can proceed in a similar fashion, use the identity $\cos^2 A = 1-\sin^2 A$ and the substitution $u = \sin x$.

Example

To find $\int \sin^4 x\cos^3 x\,\mathrm{d}x$ we write $\int \sin^4 x(\cos^2 x\cdot\cos x)\,\mathrm{d}x$. Using the identity $\cos^2 x = 1-\sin^2 x$ this becomes

$$\begin{aligned}\int \sin^4 x(\cos^2 x\cdot\cos x)\,\mathrm{d}x &= \int \sin^4 x(1-\sin^2 x)\cos x\,\mathrm{d}x \\ &= \int (\sin^4 x\cos x-\sin^6 x\cos x)\,\mathrm{d}x\end{aligned}$$

Then the substitution $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$ gives

$$\begin{aligned}\int (u^4-u^6)\mathrm{d}u &= \frac{u^5}{5}-\frac{u^7}{7}+c \\ &= \frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+c\end{aligned}$$

In the case when both m and n are even you should try using the double angle formulae, as in Exercise 2 Q2 below.

Exercises 2

1. (a) Find $\int \cos^3 x\,\mathrm{d}x$ (b) $\int \cos^5 x\,\mathrm{d}x$ (c) $\int \sin^5 x\cos^2 x\,\mathrm{d}x$.

2. Evaluate $\int \sin^2 x\cos^2 x\mathrm{d}x$ by using the double angle formulae

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

3. Using the double angle formulae twice find $\int \sin^4 x\cos^2 x\,\mathrm{d}x$.

4. Integrals which make use of a trigonometric substitution

There are several integrals which can be found by making a trigonometric substitution. Consider the following example.

Example

Suppose we wish to find $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x$.

Let us see what happens when we make the substitution $x = \tan\theta$.

Our reason for doing this is that the integrand will then involve $\dfrac{1}{1+\tan^2\theta}$ and we have an identity $(1+\tan^2 A = \sec^2 A)$ which will enable us to simplify this.

With $x = \tan\theta$, $\dfrac{dx}{d\theta} = \sec^2\theta$, so that $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. The integral becomes

$$\begin{aligned}
\int \frac{1}{1+x^2}\mathrm{d}x &= \int \frac{1}{1+\tan^2\theta}\sec^2\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{\sec^2\theta}\sec^2\theta\,\mathrm{d}\theta \\
&= \int 1\,\mathrm{d}\theta \\
&= \theta + c \\
&= \tan^{-1} x + c
\end{aligned}$$

So $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1} x + c$. This is an important standard result.

We can generalise this result to the integral $\displaystyle\int \frac{1}{a^2+x^2}\mathrm{d}x$:

We make the substitution $x = a\tan\theta$, $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$. The integral becomes

$$\int \frac{1}{a^2+a^2\tan^2\theta} a\sec^2\theta\,\mathrm{d}\theta$$

and using the identity $1+\tan^2\theta = \sec^2\theta$ this reduces to

$$\begin{aligned}
\frac{1}{a}\int 1\,\mathrm{d}\theta &= \frac{1}{a}\theta + c \\
&= \frac{1}{a}\tan^{-1}\frac{x}{a} + c
\end{aligned}$$

This is a standard result which you should be aware of and be prepared to look up when necessary.

Key Point

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1} x + c \qquad\qquad \int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

Example

Suppose we seek $\displaystyle\int \frac{1}{4+9x^2}\,\mathrm{d}x$.

We proceed by first extracting a factor of 4 from the denominator:

$$\int \frac{1}{4+9x^2}\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{1+\frac{9}{4}x^2}\,\mathrm{d}x$$

This is very close to the standard result in the previous keypoint except that the term $\frac{9}{4}$ is not really wanted. Let us observe the effect of making the substitution $u = \frac{3}{2}x$, so that $u^2 = \frac{9}{4}x^2$. Then $\mathrm{d}u = \frac{3}{2}\mathrm{d}x$ and the integral becomes

$$\begin{aligned}\frac{1}{4}\int \frac{1}{1+\frac{9}{4}x^2}\,\mathrm{d}x &= = \frac{1}{4}\int \frac{1}{1+u^2}\cdot\frac{2}{3}\,\mathrm{d}u \\ &= \frac{1}{6}\int \frac{1}{1+u^2}\,\mathrm{d}u\end{aligned}$$

This can be finished off using the standard result, to give $\displaystyle\frac{1}{6}\tan^{-1}u + c = \frac{1}{6}\tan^{-1}\frac{3}{2}x + c$.

We now consider a similar example for which a sine substitution is appropriate.

Example

Suppose we wish to find $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x$.

The substitution we will use here is based upon the observations that in the denominator we have a term $a^2 - x^2$, and that there is a trigonometric identity $1 - \sin^2 A = \cos^2 A$ (and hence $(a^2 - a^2\sin^2 A = a^2\cos^2 A)$.

We try $x = a\sin\theta$, so that $x^2 = a^2\sin^2\theta$. Then $\frac{dx}{d\theta} = a\cos\theta$ and $\mathrm{d}x = a\cos\theta\mathrm{d}\theta$. The integral becomes

$$\begin{aligned}\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x &= \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}a\cos\theta\mathrm{d}\theta \\ &= \int \frac{1}{\sqrt{a^2\cos^2\theta}}a\cos\theta\mathrm{d}\theta \\ &= \int \frac{1}{a\cos\theta}a\cos\theta\mathrm{d}\theta \\ &= \int 1\,\mathrm{d}\theta \\ &= \theta + c \\ &= \sin^{-1}\frac{x}{a} + c\end{aligned}$$

Hence $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \sin^{-1}\frac{x}{a} + c$.
This is another standard result.

Key Point

$$\int \frac{1}{\sqrt{a^2 - x^2}}\mathrm{d}x = \sin^{-1}\frac{x}{a} + c$$

Example

Suppose we wish to find $\displaystyle\int \frac{1}{\sqrt{4 - 9x^2}}\mathrm{d}x$.

The trick is to try to write this in the standard form. Let $u = 3x$, $\mathrm{d}u = 3\mathrm{d}x$ so that

$$\begin{aligned}\int \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x &= \frac{1}{3}\int \frac{1}{\sqrt{4-u^2}}\,\mathrm{d}u \\ &= \frac{1}{3}\sin^{-1}\frac{u}{2} + c \\ &= \frac{1}{3}\sin^{-1}\frac{3x}{2} + c\end{aligned}$$

Exercises 3

1. Use the trigonometric substitution indicated to find the given integral.

(a) $\displaystyle\int \frac{x^2}{\sqrt{16 - x^2}}\mathrm{d}x$ let $x = 4\sin\theta$ (b) $\displaystyle\int \frac{1}{1 + 4x^2}\mathrm{d}x$ let $x = \frac{1}{2}\tan\theta$.

Answers

Exercises 1

1. (a) $\dfrac{x}{2} + \dfrac{1}{4}\sin 2x + c$ (b) $\frac{\pi}{4}$ (c) $-\dfrac{\cos 4x}{8} + c$

2. (a) $\dfrac{\sqrt{3}}{8} = 2.165$ (3 d.p.) (b) $t + c$

(c) $\dfrac{1}{6}\sin 3t - \dfrac{1}{22}\sin 11t + c$

Exercises 2

1. (a) $\frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x + c$ (b) $\frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x + c$
   (c) $-\frac{1}{7}\sin^4 x\cos^3 x - \frac{4}{35}\sin^2 x\cos^3 x - \frac{8}{105}\cos^3 x + c$.

2. $-\frac{1}{4}\sin x\cos^3 x + \frac{1}{8}\cos x\sin x + \frac{1}{8}x + c$.

3. $-\frac{1}{6}\sin^3 x\cos^3 x - \frac{1}{8}\sin x\cos^3 x + \frac{1}{16}\cos x\sin x + \frac{1}{16}x + c$

Exercises 3

1. (a) $-\frac{1}{2}x\sqrt{16 - x^2} + 8\sin^{-1}\frac{x}{4} + c$ (b) $\frac{1}{2}\tan^{-1}2x + c$.