mathcentre

Integration using a table of anti-derivatives

We may regard integration as the reverse of differentiation. So if we have a table of derivatives, we can read it backwards as a table of anti-derivatives. When we do this, we often need to deal with constants which arise in the process of differentiation.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- use a table of derivatives, or a table of anti-derivatives, in order to integrate simple functions.

Contents

1. Introduction

When we are integrating, we need to be able to recognise standard forms. The following table gives a list of standard forms, obtained as anti-derivatives. Sometimes, it may be possible to use one of these standard forms directly. On other occasions, some manipulation will be needed first.

Key Point

$$
\begin{aligned}
\int x^n \mathrm{d}x &= \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1) \\
\int (ax+b)^n \mathrm{d}x &= \frac{(ax+b)^{n+1}}{a(n+1)} + c \qquad (n \neq -1) \\
\int \frac{1}{x}\,\mathrm{d}x &= \ln|x| + c \\
\int \frac{1}{ax+b}\,\mathrm{d}x &= \frac{1}{a}\ln|ax+b| + c \\
\int \mathrm{e}^x \mathrm{d}x &= \mathrm{e}^x + c \\
\int \mathrm{e}^{mx} \mathrm{d}x &= \frac{1}{m}\mathrm{e}^{mx} + c \\
\int \cos x\,\mathrm{d}x &= \sin x + c \\
\int \cos nx\,\mathrm{d}x &= \frac{1}{n}\sin nx + c \\
\int \sin x\,\mathrm{d}x &= -\cos nx + c \\
\int \sin nx\,\mathrm{d}x &= -\frac{1}{n}\cos nx + c \\
\int \sec^2 x\,\mathrm{d}x &= \tan x + c \\
\int \sec^2 nx\,\mathrm{d}x &= \frac{1}{n}\tan nx + c \\
\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \sin^{-1} x + c \\
\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x &= \sin^{-1}\left(\frac{x}{a}\right) + c \\
\int \frac{1}{1+x^2}\,\mathrm{d}x &= \tan^{-1} x + c \\
\int \frac{1}{a^2+x^2}\,\mathrm{d}x &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c
\end{aligned}
$$

2. Integrating powers

We know that the derivative of x^n is nx^{n-1}. Replacing n by $n+1$ we see that the derivative of x^{n+1} is $(n+1)x^n$, so that the derivative of $\dfrac{x^{n+1}}{n+1}$ is x^n (provided that $n+1 \neq 0$). Thus

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c\,.$$

Similarly, the derivative of $(ax+b)^n$ is $an(ax+b)^{n-1}$. Replacing n by $n+1$ we see that the derivative of $(ax+b)^{n+1}$ is $a(n+1)(ax+b)^n$, so that the derivative of $\dfrac{(ax+b)^{n+1}}{a(n+1)}$ is $(ax+b)^n$ (provided that $n+1 \neq 0$ and that $a \neq 0$). Thus

$$\int (ax+b)^n \mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + c\,.$$

What happens if $n = -1$, so that $n+1 = 0$? We know that the derivative of $\ln|x|$ is $1/x$, so that

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c\,.$$

Simlarly, the derivative of $\ln|ax+b|$ is $\dfrac{a}{ax+b}$, so that the derivative of $\dfrac{1}{a}\ln|ax+b|$ is $\dfrac{1}{ax+b}$. Thus

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c\,.$$

Example

Find $\displaystyle\int \frac{1}{2-3x}\,\mathrm{d}x$.

Here, $a = -3$ and $b = 2$, so

$$\int \frac{1}{2-3x}\,\mathrm{d}x = -\frac{1}{3}\ln|2-3x| + c\,.$$

3. Integrating exponentials

We know that the derivative of e^x remains unchanged, as e^x. Thus

$$\int \mathrm{e}^x \mathrm{d}x = \mathrm{e}^x + c\,.$$

Similarly, we know that the derivative of e^{mx} is $m\mathrm{e}^{mx}$, so that the derivative of $\dfrac{1}{m}\mathrm{e}^{mx}$ is e^{mx}. Thus

$$\int \mathrm{e}^{mx} \mathrm{d}x = \frac{1}{m}\mathrm{e}^{mx} + c\,.$$

Example

Find $\int e^{4x} dx$.

Here, $m = 4$, so
$$\int e^{4x} dx = \frac{1}{4} e^4 x + c\,.$$

4. Integrating trigonometric functions

We know that the derivative of $\sin x$ is $\cos x$. Thus
$$\int \cos x \, dx = \sin x + c\,.$$

Similarly, we know that the derivative of $\sin nx$ is $n \cos nx$, so that the derivative of $\frac{1}{n} \sin nx$ is $\cos nx$. Thus
$$\int \cos nx \, dx = \frac{1}{n} \sin nx + c\,.$$
We also know that the derivative of $\cos x$ is $-\sin x$. Thus
$$\int \sin x \, dx = -\cos x + c\,.$$

Similarly, we know that the derivative of $\cos nx$ is $-n \sin nx$, so that the derivative of $-\frac{1}{n} \cos nx$ is $\sin nx$. Thus
$$\int \sin nx \, dx = -\frac{1}{n} \cos nx + c\,.$$

We can use the fact that $\tan x = \dfrac{\sin x}{\cos x}$ to find an anti-derivative of $\tan x$. We use the rule for logarithmic differentiation to see that the derivative of $\ln|\cos x|$ is $\dfrac{-\sin x}{\cos x}$, so that
$$\begin{aligned} \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= -\ln|\cos x| + c \\ &= \ln|\sec x| + c\,. \end{aligned}$$
(In the last step of this argument, we have used the fact that $-\ln u$ is equal to $\ln(1/u)$.)

There is one more trigonometric function which we can integrate without difficulty. We know that the derivative of $\tan x$ is $\sec^2 x$. Thus
$$\int \sec^2 x \, dx = \tan x + c\,.$$

Similarly, the derivative of $\tan nx$ is $n \sec^2 nx$, so that the derivative of $\frac{1}{n} \tan nx$ is $\sec^2 nx$. Thus
$$\int \sec^2 nx \, dx = \frac{1}{n} \tan nx + c$$

5. Integrals giving rise to inverse trigonometric functions

Sometimes, integrals involving fractions and square roots give rise to inverse trigonometric functions.

We know that the derivative of $\sin^{-1} x$ is $\dfrac{1}{\sqrt{1-x^2}}$. Thus

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1} x + c\,.$$

Similarly, we know that the derivative of $\sin^{-1}\left(\dfrac{x}{a}\right)$ is $\dfrac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}$, which equals $\dfrac{1}{\sqrt{a^2-x^2}}$. Thus

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + c\,.$$

Example

Find $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x$.

Here, $a = \sqrt{4} = 2$, so that

$$\int \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{2}\right) + c\,.$$

Example

Find $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$.

This is not quite in our standard form. However, we can take the 9 outside the square root, so that it becomes 3. We get

$$\int \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x = \int \frac{1}{3}\cdot\frac{1}{\sqrt{\frac{4}{9}-x^2}}\,\mathrm{d}x\,,$$

and this is in the standard form. So now we can take the $\frac{1}{3}$ outside the integral, and we see that $a = \sqrt{\frac{4}{9}} = \frac{2}{3}$, so that

$$\begin{aligned}
\int \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x &= \frac{1}{3}\int \frac{1}{\sqrt{\frac{4}{9}-x^2}}\,\mathrm{d}x \\
&= \frac{1}{3}\sin^{-1}\left(\frac{x}{\frac{2}{3}}\right) + c \\
&= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c\,.
\end{aligned}$$

Another type of integral which may be found using an inverse trigonometric function involves a fraction, but does not involve a square root.

We know that the derivative of $\tan^{-1} x$ is $\dfrac{1}{1+x^2}$. Thus

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1} x + c\,.$$

Similarly, we know that the derivative of $\tan^{-1}\left(\dfrac{x}{a}\right)$ is $\dfrac{1}{a(1+\left(\frac{x}{a}\right)^2)}$, which equals $\dfrac{a}{a^2+x^2}$, so that the derivative of $\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$ is $\dfrac{1}{a^2+x^2}$. Thus

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Example

Find $\displaystyle\int \frac{1}{9+x^2}\,\mathrm{d}x$.

Here, $a = \sqrt{9} = 3$, so that

$$\int \frac{1}{9+x^2}\,\mathrm{d}x = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c\,.$$

Example

Find $\displaystyle\int \frac{1}{25+16x^2}\,\mathrm{d}x$.

Here, we take the 16 outside the integral, so that we get

$$\int \frac{1}{25+16x^2}\,\mathrm{d}x = \frac{1}{16}\int \frac{1}{\frac{25}{16}+x^2}\,\mathrm{d}x\,.$$

Now we can see that $a = \sqrt{\frac{25}{16}} = \frac{5}{4}$, so that

$$\begin{aligned}\int \frac{1}{25+16x^2}\,\mathrm{d}x &= \frac{1}{16}\times\frac{1}{\left(\frac{5}{4}\right)}\tan^{-1}\left(\frac{x}{\left(\frac{5}{4}\right)}\right) + c\\ &= \frac{1}{16}\times\frac{4}{5}\tan^{-1}\left(\frac{4x}{5}\right) + c\\ &= \frac{1}{20}\tan^{-1}\left(\frac{4x}{5}\right) + c\,.\end{aligned}$$

Exercises

1. Determine the integral of each of the following functions

(a)	x^8	(b)	$\dfrac{1}{x^3}$	(c)	$\dfrac{1}{\sqrt{x}}$	(d)	$\dfrac{1}{x}$	(e)	$\sin 5x$
(f)	$\sec^2 2x$	(g)	$\dfrac{1}{16+x^2}$	(h)	$\dfrac{1}{\sqrt{4-x^2}}$	(i)	$\dfrac{1}{\sqrt{16-9x^2}}$	(j)	$\dfrac{1}{4+25x^2}$

2. Integration has the same linearity rules as differentiation, namely

$$\int kf(x)\,dx = k\int f(x)\,dx \qquad \text{and} \qquad \int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx\,.$$

Use these rules to determine the integrals of the following functions

(a) $5x^4 + 10\cos 2x$ (b) $12\mathrm{e}^{4x} + 4\sqrt{x}$ (c) $36\sec^2 4x + 12\mathrm{e}^{-3x}$

(d) $12x^6 - 2\sin 4x$ (e) $\dfrac{54}{9+x^2} + \dfrac{15}{\sqrt{9-x^2}}$ (f) $10\cos 5x - 5\cos 10x$

Answers

1. In all answers the constant of integration has been omitted.

(a) $\dfrac{1}{9}x^9$ (b) $-\dfrac{1}{2}x^{-2} = -\dfrac{1}{2x^2}$ (c) $2x^{1/2} = 2\sqrt{x}$ (d) $\ln x$

(e) $-\dfrac{1}{5}\cos 5x$ (f) $\dfrac{1}{2}\tan 2x$ (g) $\dfrac{1}{4}\tan^{-1}\left(\dfrac{x}{4}\right)$ (h) $\sin^{-1}\left(\dfrac{x}{2}\right)$

(i) $\dfrac{1}{3}\sin^{-1}\left(\dfrac{3x}{4}\right)$ (j) $\dfrac{1}{10}\tan^{-1}\left(\dfrac{5x}{2}\right)$

2. In all answers the constant of integration has been omitted.

(a) $x^5 + 5\sin 2x$ (b) $3\mathrm{e}^{4x} + \dfrac{8}{3}x^{3/2}$ (c) $9\tan 4x - 4\mathrm{e}^{-3x}$

(d) $\dfrac{12}{7}x^7 + \dfrac{1}{2}\cos 4x$ (e) $18\tan^{-1}\left(\dfrac{x}{3}\right) + 15\sin^{-1}\left(\dfrac{x}{3}\right)$ (f) $2\sin 5x - \dfrac{1}{2}\sin 10x$