Partial Derivatives 18.2

Introduction

When a function of more than one independent input variable changes because of changes in one or more of the input variables, it is important to calculate the change in the function itself. This can be investigated by holding all but one of the variables constant and finding the rate of change of the function with respect to the one remaining variable. This process is called partial differentiation. In this Section we show how to carry out the process.

Prerequisites

Before starting this Section you should ...

- understand the principle of differentiating a function of one variable

Learning Outcomes

On completion you should be able to ...

- understand the concept of partial differentiation
- differentiate a function partially with respect to each of its variables in turn
- evaluate first partial derivatives
- carry out successive partial differentiations
- formulate second partial derivatives

1. First partial derivatives

The x partial derivative

For a function of a single variable, $y = f(x)$, changing the independent variable x leads to a corresponding change in the dependent variable y. The **rate of change** of y with respect to x is given by the **derivative**, written $\dfrac{df}{dx}$. A similar situation occurs with functions of more than one variable. For clarity we shall concentrate on functions of just two variables.

In the relation $z = f(x, y)$ the **independent** variables are x and y and the **dependent variable** z. We have seen in Section 18.1 that as x and y vary the z-value traces out a surface. Now both of the variables x and y may change *simultaneously* inducing a change in z. However, rather than consider this general situation, to begin with we shall hold one of the independent variables **fixed**. This is equivalent to moving along a curve obtained by intersecting the surface by one of the coordinate planes.

Consider $f(x, y) = x^3 + 2x^2y + y^2 + 2x + 1$.

Suppose we keep y constant and vary x; then what is the rate of change of the function f?

Suppose we hold y at the value 3 then

$$f(x, 3) = x^3 + 6x^2 + 9 + 2x + 1 = x^3 + 6x^2 + 2x + 10$$

In effect, we now have a function of x only. If we differentiate it with respect to x we obtain the expression:

$$3x^2 + 12x + 2.$$

We say that f has been **partially differentiated** with respect to x. We denote the partial derivative of f with respect to x by $\dfrac{\partial f}{\partial x}$ (to be read as 'partial dee f by dee x'). In this example, when $y = 3$:

$$\frac{\partial f}{\partial x} = 3x^2 + 12x + 2.$$

In the same way if y is held at the value 4 then $f(x, 4) = x^3 + 8x^2 + 16 + 2x + 1 = x^3 + 8x^2 + 2x + 17$ and so, for this value of y

$$\frac{\partial f}{\partial x} = 3x^2 + 16x + 2.$$

Now if we return to the original formulation

$$f(x, y) = x^3 + 2x^2y + y^2 + 2x + 1$$

and treat y as a constant then the process of partial differentiation with respect to x gives

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 + 4xy + 0 + 2 + 0 \\ &= 3x^2 + 4xy + 2. \end{aligned}$$

Key Point 1

The Partial Derivative of f with respect to x

For a function of two variables $z = f(x, y)$ the partial derivative of f with respect to x is denoted by $\dfrac{\partial f}{\partial x}$ and is obtained by differentiating $f(x, y)$ with respect to x in the usual way but treating the y-variable as if it were a constant.

Alternative notations for $\dfrac{\partial f}{\partial x}$ are $f_x(x, y)$ or f_x or $\dfrac{\partial z}{\partial x}$.

Example 2

Find $\dfrac{\partial f}{\partial x}$ for (a) $f(x, y) = x^3 + x + y^2 + y$, (b) $f(x, y) = x^2y + xy^3$.

Solution

(a) $\dfrac{\partial f}{\partial x} = 3x^2 + 1 + 0 + 0 = 3x^2 + 1$ (b) $\dfrac{\partial f}{\partial x} = 2x \times y + 1 \times y^3 = 2xy + y^3$

The y partial derivative

For functions of two variables $f(x, y)$ the x and y variables are on the same footing, so what we have done for the x-variable we can do for the y-variable. We can thus imagine keeping the x-variable fixed and determining the rate of change of f as y changes. This rate of change is denoted by $\dfrac{\partial f}{\partial y}$.

Key Point 2

The Partial Derivative of f with respect to y

For a function of two variables $z = f(x, y)$ the partial derivative of f with respect to y is denoted by $\dfrac{\partial f}{\partial y}$ and is obtained by differentiating $f(x, y)$ with respect to y in the usual way but treating the x-variable as if it were a constant.

Alternative notations for $\dfrac{\partial f}{\partial y}$ are $f_y(x, y)$ or f_y or $\dfrac{\partial z}{\partial y}$.

Returning to $f(x,y) = x^3 + 2x^2y + y^2 + 2x + 1$ once again, we therefore obtain:

$$\frac{\partial f}{\partial y} = 0 + 2x^2 \times 1 + 2y + 0 + 0 = 2x^2 + 2y.$$

Example 3

Find $\dfrac{\partial f}{\partial y}$ for (a) $f(x,y) = x^3 + x + y^2 + y$ (b) $f(x,y) = x^2y + xy^3$

Solution

(a) $\dfrac{\partial f}{\partial y} = 0 + 0 + 2y + 1 = 2y + 1$ (b) $\dfrac{\partial f}{\partial y} = x^2 \times 1 + x \times 3y^2 = x^2 + 3xy^2$

We can calculate the partial derivative of f with respect to x and the value of $\dfrac{\partial f}{\partial x}$ at a specific point e.g. $x = 1,\ y = -2$.

Example 4

Find $f_x(1,-2)$ and $f_y(-3,2)$ for $f(x,y) = x^2 + y^3 + 2xy$.

[Remember f_x means $\dfrac{\partial f}{\partial x}$ and f_y means $\dfrac{\partial f}{\partial y}$.]

Solution

$f_x(x,y) = 2x + 2y$, so $f_x(1,-2) = 2 - 4 = -2$; $f_y(x,y) = 3y^2 + 2x$, so $f_y(-3,2) = 12 - 6 = 6$

Task

Given $f(x,y) = 3x^2 + 2y^2 + xy^3$ find $f_x(1,-2)$ and $f_y(-1,-1)$.

First find expressions for $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$:

Your solution

$\dfrac{\partial f}{\partial x} =$ $\dfrac{\partial f}{\partial y} =$

Answer

$\dfrac{\partial f}{\partial x} = 6x + y^3, \quad \dfrac{\partial f}{\partial y} = 4y + 3xy^2$

Now calculate $f_x(1,-2)$ and $f_y(-1,-1)$:

Your solution

$f_x(1,-2) =$ $\qquad\qquad$ $f_y(-1,-1) =$

Answer

$f_x(1,-2) = 6 \times 1 + (-2)^3 = -2;$ $\qquad f_y(-1,-1) = 4 \times (-1) + 3(-1) \times 1 = -7$

Functions of several variables

As we have seen, a function of two variables $f(x,y)$ has two partial derivatives, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. In an exactly analogous way a function of three variables $f(x,y,u)$ has three partial derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial u}$, and so on for functions of more than three variables. Each partial derivative is obtained in the same way as stated in Key Point 3:

Key Point 3

The Partial Derivatives of $\boldsymbol{f(x, y, u, v, w, \ldots)}$

For a function of several variables $z = f(x,y,u,v,w,\ldots)$ the partial derivative of f with respect to v (say) is denoted by $\dfrac{\partial f}{\partial v}$ and is obtained by differentiating $f(x,y,u,v,w,\ldots)$ with respect to v in the usual way but treating all the other variables as if they were constants.

Alternative notations for $\dfrac{\partial f}{\partial v}$ when $z = f(x,y,u,v,w,\ldots)$ are $f_v(x,y,u,v,w\ldots)$ and f_v and $\dfrac{\partial z}{\partial v}$.

Task

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial u}$ for $f(x,y,u,v) = x^2 + xy^2 + y^2u^3 - 7uv^4$

Your solution

$\dfrac{\partial f}{\partial x} =$ $\qquad\qquad$ $\dfrac{\partial f}{\partial u} =$

Answer

$\dfrac{\partial f}{\partial x} = 2x + y^2 + 0 + 0 = 2x + y^2;$ $\quad \dfrac{\partial f}{\partial u} = 0 + 0 + y^2 \times 3u^2 - 7v^4 = 3y^2u^2 - 7v^4.$

Task

The pressure, P, for one mole of an ideal gas is related to its absolute temperature, T, and specific volume, v, by the equation

$$Pv = RT$$

where R is the gas constant.

Obtain simple expressions for

(a) the coefficient of thermal expansion, α, defined by:

$$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P$$

(b) the isothermal compressibility, κ_T, defined by:

$$\kappa_T = -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T$$

Your solution

(a)

Answer

$$v = \frac{RT}{P} \quad \Rightarrow \quad \left(\frac{\partial v}{\partial T}\right)_P = \frac{R}{P}$$

so $$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P = \frac{R}{Pv} = \frac{1}{T}$$

Your solution

(b)

Answer

$$v = \frac{RT}{P} \quad \Rightarrow \quad \left(\frac{\partial v}{\partial P}\right)_T = -\frac{RT}{P^2}$$

so $$\kappa_T = -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T = \frac{RT}{vP^2} = \frac{1}{P}$$

Exercises

1. For the following functions find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$

   (a) $f(x, y) = x + 2y + 3$

   (b) $f(x, y) = x^2 + y^2$

   (c) $f(x, y) = x^3 + xy + y^3$

   (d) $f(x, y) = x^4 + xy^3 + 2x^3y^2$

   (e) $f(x, y, z) = xy + yz$

2. For the functions of Exercise 1 (a) to (d) find $f_x(1, 1),\ f_x(-1, -1),\ f_y(1, 2),\ f_y(2, 1)$.

Answers

1. (a) $\dfrac{\partial f}{\partial x} = 1, \quad \dfrac{\partial f}{\partial y} = 2$

   (b) $\dfrac{\partial f}{\partial x} = 2x, \quad \dfrac{\partial f}{\partial y} = 2y$

   (c) $\dfrac{\partial f}{\partial x} = 3x^2 + y, \quad \dfrac{\partial f}{\partial y} = x + 3y^2$

   (d) $\dfrac{\partial f}{\partial x} = 4x^3 + y^3 + 6x^2y^2, \quad \dfrac{\partial f}{\partial y} = 3xy^2 + 4x^3y$

   (e) $\dfrac{\partial f}{\partial x} = y, \quad \dfrac{\partial f}{\partial y} = x + z$

2.

	$f_x(1,1)$	$f_x(-1,-1)$	$f_y(1,2)$	$f_y(2,1)$
(a)	1	1	2	2
(b)	2	-2	4	2
(c)	4	2	13	5
(d)	11	1	20	38

2. Second partial derivatives

Performing two successive partial differentiations of $f(x, y)$ with respect to x (holding y constant) is denoted by $\dfrac{\partial^2 f}{\partial x^2}$ (or $f_{xx}(x, y)$) and is defined by

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

For functions of two or more variables as well as $\dfrac{\partial^2 f}{\partial x^2}$ other second-order partial derivatives can be obtained. Most obvious is the second derivative of $f(x, y)$ with respect to y is denoted by $\dfrac{\partial^2 f}{\partial y^2}$ (or $f_{yy}(x, y)$) which is defined as:

$$\frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Example 5

Find $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ for $f(x, y) = x^3 + x^2y^2 + 2y^3 + 2x + y$.

Solution

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2 + 0 + 2 + 0 = 3x^2 + 2xy^2 + 2$$

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x + 2y^2 + 0 = 6x + 2y^2.$$

$$\frac{\partial f}{\partial y} = 0 + x^2 \times 2y + 6y^2 + 0 + 1 = 2x^2y + 6y^2 + 1$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2x^2 + 12y.$$

We can use the alternative notation when evaluating derivatives.

Example 6

Find $f_{xx}(-1, 1)$ and $f_{yy}(2, -2)$ for $f(x, y) = x^3 + x^2y^2 + 2y^3 + 2x + y$.

Solution

$f_{xx}(-1, 1) = 6 \times (-1) + 2 \times (-1)^2 = -4.$

$f_{yy}(2, -2) = 2 \times (2)^2 + 12 \times (-2) = -16$

Mixed second derivatives

It is possible to carry out a partial differentiation of $f(x,y)$ with respect to x followed by a partial differentiation with respect to y (or vice-versa). The results are examples of **mixed derivatives**. We must be careful with the notation here.

We use $\dfrac{\partial^2 f}{\partial x \partial y}$ to mean 'differentiate first with respect to y and then with respect to x' and we use $\dfrac{\partial^2 f}{\partial y \partial x}$ to mean 'differentiate first with respect to x and then with respect to y':

$$\text{i.e.} \qquad \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

(This explains why the order is opposite of what we expect - the derivative 'operates on the left'.)

Example 7

For $f(x,y) = x^3 + 2x^2y^2 + y^3$ find $\dfrac{\partial^2 f}{\partial x \partial y}$.

Solution

$\dfrac{\partial f}{\partial y} = 4x^2y + 3y^2; \quad \dfrac{\partial^2 f}{\partial x \partial y} = 8xy$

The remaining possibility is to differentiate first with respect to x and then with respect to y i.e. $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$.

For the function in Example 7 $\dfrac{\partial f}{\partial x} = 3x^2 + 4xy^2$ and $\dfrac{\partial^2 f}{\partial y \partial x} = 8xy$. Notice that for this function

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial^2 f}{\partial y \partial x}.$$

This equality of mixed derivatives is true for all functions which you are likely to meet in your studies.

To evaluate a mixed derivative we can use the alternative notation. To evaluate $\dfrac{\partial^2 f}{\partial x \partial y}$ we write $f_{yx}(x,y)$ to indicate that the first differentiation is with respect to y. Similarly, $\dfrac{\partial^2 f}{\partial y \partial x}$ is denoted by $f_{xy}(x,y)$.

Example 8

Find $f_{yx}(1,2)$ for the function $f(x,y) = x^3 + 2x^2y^2 + y^3$

Solution

$$f_x = 3x^2 + 4xy^2 \quad \text{and} \quad f_{yx} = 8xy \qquad \text{so} \quad f_{yx}(1,2) = 8 \times 1 \times 2 = 16.$$

Task

Find $f_{xx}(1,2)$, $f_{yy}(-2,-1)$, $f_{xy}(3,3)$ for $f(x,y) \equiv x^3 + 3x^2y^2 + y^2$.

Your solution

Answer

$f_x = 3x^2 + 6xy^2$; $\quad f_y = 6x^2y + 2y$

$f_{xx} = 6x + 6y^2$; $\quad f_{yy} = 6x^2 + 2$; $\quad f_{xy} = f_{yx} = 12xy$

$f_{xx}(1,2) = 6 + 24 = 30$; $\quad f_{yy}(-2,-1) = 26$; $\quad f_{xy}(3,3) = 108$

Engineering Example 1

The ideal gas law and Redlich-Kwong equation

Introduction

In Chemical Engineering it is often necessary to be able to equate the pressure, volume and temperature of a gas. One relevant equation is the ideal gas law

$$PV = nRT \qquad (1)$$

where P is pressure, V is volume, n is the number of moles of gas, T is temperature and R is the ideal gas constant ($= 8.314$ J mol^{-1} K^{-1}, when all quantities are in S.I. units). The ideal gas law has been in use since 1834, although its special cases at constant temperature (Boyle's Law, 1662) and constant pressure (Charles' Law, 1787) had been in use many decades previously.

While the ideal gas law is adequate in many circumstances, it has been superseded by many other laws where, in general, simplicity is weighed against accuracy. One such law is the Redlich-Kwong equation

$$P = \frac{RT}{V-b} - \frac{a}{\sqrt{T}\,V(V+b)} \qquad (2)$$

where, in addition to the variables in the ideal gas law, the extra parameters a and b are dependent upon the particular gas under consideration.

Clearly, in both equations the temperature, pressure and volume will be positive. Additionally, the Redlich-Kwong equation is only valid for values of volume greater than the parameter b - in practice however, this is not a limitation, since the gas would condense to a liquid before this point was reached.

Problem in words

Show that for both Equations (1) and (2)

(a) for constant temperature, the pressure decreases as the volume increases

(Note : in the Redlich-Kwong equation, assume that T is large.)

(b) for constant volume, the pressure increases as the temperature increases.

Mathematical statement of problem

For both Equations (1) and (2), and for the allowed ranges of the variables, show that

(a) $\dfrac{\partial P}{\partial V} < 0$ for $T =$ constant

(b) $\dfrac{\partial P}{\partial T} > 0$ for $V =$ constant

Assume that T is sufficiently large so that terms in $T^{-1/2}$ may be neglected when compared to terms in T.

Mathematical analysis

1. **Ideal gas law**
   This can be rearranged as

$$P = \frac{nRT}{V}$$

   so that

   (i) at constant temperature

$$\frac{\partial P}{\partial V} = \frac{-nRT}{V^2} \quad < 0 \quad \text{as all quantities are positive}$$

   (ii) for constant volume

$$\frac{\partial P}{\partial T} = \frac{nR}{V} \quad > 0 \quad \text{as all quantities are positive}$$

2. **Redlich-Kwong equation**

$$\begin{aligned} P &= \frac{RT}{V-b} - \frac{a}{\sqrt{T}\,V(V+b)} \\ &= RT(V-b)^{-1} - a\,T^{-1/2}\,(V^2+Vb)^{-1} \end{aligned}$$

   so that

   (i) at constant temperature

$$\frac{\partial P}{\partial V} = -RT(V-b)^{-2} + a\,T^{-1/2}(V^2+Vb)^{-2}(2V+b)$$

   which, for large T, can be approximated by

$$\frac{\partial P}{\partial V} \approx \frac{-RT}{(V-b)^2} \quad < 0 \quad \text{as all quantities are positive}$$

   (ii) for constant volume

$$\frac{\partial P}{\partial T} = R(V-b)^{-1} + \frac{1}{2}\,a\,T^{-3/2}(V^2+Vb)^{-1} \quad > 0 \quad \text{as all quantities are positive}$$

Interpretation

In practice, the restriction on T is not severe, and regions in which $\dfrac{\partial P}{\partial V} < 0$ does not apply are those in which the gas is close to liquefying and, therefore, the entire Redlich-Kwong equation no longer applies.

Exercises

1. For the following functions find $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$.

   (a) $f(x, y) = x + 2y + 3$

   (b) $f(x, y) = x^2 + y^2$

   (c) $f(x, y) = x^3 + xy + y^3$

   (d) $f(x, y) = x^4 + xy^3 + 2x^3y^2$

   (e) $f(x, y, z) = xy + yz$

2. For the functions of Exercise 1 (a) to (d) find $f_{xx}(1, -3), \quad f_{yy}(-2, -2), \quad f_{xy}(-1, 1)$.

3. For the following functions find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial t}$

   (a) $f(x, t) = x \sin(tx) + x^2 t$ (b) $f(x, t, z) = zxt - e^{xt}$ (c) $f(x, t) = 3\cos(t + x^2)$

Answers

1. (a) $\dfrac{\partial^2 f}{\partial x^2} = 0 = \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

   (b) $\dfrac{\partial^2 f}{\partial x^2} = 2 = \dfrac{\partial^2 f}{\partial y^2}; \quad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = 0$

   (c) $\dfrac{\partial^2 f}{\partial x^2} = 6x, \quad \dfrac{\partial^2 f}{\partial y^2} = 6y; \quad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = 1.$

   (d) $\dfrac{\partial^2 f}{\partial x^2} = 12x^2 + 12xy^2, \quad \dfrac{\partial^2 f}{\partial y^2} = 6xy + 4x^3, \quad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = 3y^2 + 12x^2y$

   (e) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial y^2} = 0; \quad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = 1$

2.

	$f_{xx}(1, -3)$	$f_{yy}(-2, -2)$	$f_{xy}(-1, 1)$
(a)	0	0	0
(b)	2	2	0
(c)	6	-12	1
(d)	120	-8	15

3. (a) $\dfrac{\partial f}{\partial x} = \sin(tx) + xt\cos(tx) + 2xt \qquad \dfrac{\partial^2 f}{\partial t \partial x} = \dfrac{\partial^2 f}{\partial x \partial t} = 2x\cos(tx) - x^2 t \sin(tx) + 2x$

   (b) $\dfrac{\partial f}{\partial x} = zt - te^{xt} \qquad \dfrac{\partial^2 f}{\partial t \partial x} = \dfrac{\partial^2 f}{\partial x \partial t} = z - e^{xt} - txe^{xt}$

   (c) $\dfrac{\partial f}{\partial x} = -6x\sin(t + x^2) \qquad \dfrac{\partial^2 f}{\partial t \partial x} = \dfrac{\partial^2 f}{\partial x \partial t} = -6x\cos(t + x^2)$