The Product Rule

A special rule, **the product rule**, exists for differentiating products of two (or more) functions. This unit illustrates this rule.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- state the product rule
- differentiate products of functions

Contents

1. Introduction

Sometimes we are given functions that are actually **products** of other functions. This means, two functions multiplied together. So, an example would be

$$y = x^2 \cos 3x$$

So here we have one function, x^2, multiplied by a second function, $\cos 3x$.
Notice that we can write this as $y = uv$ where $u = x^2$ and $v = \cos 3x$.

There is a formula we can use to differentiate a product - it is called **the product rule**. In this unit we will state and use this rule.

2. The product rule

The rule states:

Key Point

The product rule: if $y = uv$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

So, when we have a product to differentiate we can use this formula.

Example:

Suppose we want to differentiate $y = x^2 \cos 3x$.

We identify u as x^2 and v as $\cos 3x$.

$$u = x^2 \qquad v = \cos 3x$$

We now write down the derivatives of each of these functions.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = -3\sin 3x$$

We now put all these results into the given formula:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= x^2 \times (-3\sin 3x) \ + \ \cos 3x \times 2x \end{aligned}$$

This can be tidied up, and common factors should be identified. Notice that in both terms there is a factor of x. This common factor is brought out in front of a bracket. So, finally

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(-3x\sin 3x + 2\cos 3x)$$

We have finished, and obtained the derivative of the product in a nice, tidy, factorised form.

Example

Suppose we want to differentiate $y = (1 - x^3)\mathrm{e}^{2x}$.

We identify u as $1 - x^3$ and v as e^{2x}. We now write down the derivatives.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -3x^2 \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 2\mathrm{e}^{2x}$$

We now put all these results into the given formula:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (1 - x^3) \times 2\mathrm{e}^{2x} + \mathrm{e}^{2x} \times (-3x^2) \\
&= \mathrm{e}^{2x}(2 - 3x^2 - 2x^3)
\end{aligned}$$

We have finished, and obtained the derivative of the product in a nice, tidy, factorised form.

Example

Suppose we want to differentiate $y = x^3(4 - x)^{1/2}$.

Identify u as x^3 and v as $(4 - x)^{1/2}$.

$$u = x^3 \qquad v = (4 - x)^{1/2}$$

Differentiating

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2 \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = -1 \cdot \frac{1}{2}(4 - x)^{-1/2}$$

To obtain the derivative of v we have used the rule for differentiating a function of a function (see the unit on the Chain Rule).

Recall from our formula that if $y = uv$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$. So

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 \cdot -\frac{1}{2}(4 - x)^{-1/2} + (4 - x)^{1/2} \cdot 3x^2$$

Again, this should be tidied up. Let us look at how we might simplify it. Remember that $(4 - x)^{-1/2}$ is the same as $\frac{1}{(4-x)^{1/2}}$. So

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^3}{2(4 - x)^{1/2}} + (4 - x)^{1/2} \cdot 3x^2$$

We now put everything over a common denominator. To do this, multiply the second term, top and bottom, by $2(4 - x)^{1/2}$:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x^3}{2(4 - x)^{1/2}} + \frac{(4 - x)^{1/2} \cdot 3x^2}{1} \cdot \frac{2(4 - x)^{1/2}}{2(4 - x)^{1/2}} \\
&= \frac{-x^3 + 6x^2(4 - x)}{2(4 - x)^{1/2}} \\
&= \frac{x^2(24 - 7x)}{2(4 - x)^{1/2}}
\end{aligned}$$

Again, note that the solution is presented in a nice, tidy form so that if we need to work with it somewhere else we can.

Exercises

1. Find the derivative of each of the following:

a) $x \tan x$ b) $x^2 e^{-x}$ c) $5e^{-2x} \sin 3x$ d) $3x^{1/2} \cos 2x$

e) $2x^6(1+x)^5$ f) $x^{-2}(1+x^2)^{1/2}$ g) $xe^x \sin x$ h) $7x^{3/2}e^{-4x} \cos 2x$

Answers

1. a) $x \sec^2 x + \tan x$ b) $x(2-x)e^{-x}$

   c) $5e^{-2x}(3\cos 3x - 2\sin 3x)$ d) $\dfrac{3}{2}x^{-1/2}(\cos 2x - 4x \sin 2x)$

   e) $2x^5(1+x)^4(6+11x)$ f) $-x^{-3}((1+x^2)^{-1/2}(2+x^2)$

   g) $e^x[(1+x)\sin x + \cos x]$ h) $\dfrac{7}{2}x^{1/2}e^{-4x}(3\cos 2x - 8x \cos 2x - 4x \sin x)$