The Chain Rule

11.5

Introduction

In this Section we will see how to obtain the derivative of a composite function (often referred to as a 'function of a function'). To do this we use the **chain rule**. This rule can be used to obtain the derivatives of functions such as e^{x^2+3x} (the exponential function of a polynomial); $\sin(\ln x)$ (the sine function of the natural logarithm function); $\sqrt{x^3+4}$ (the square root function of a polynomial).

Prerequisites

Before starting this Section you should ...

- be able to differentiate standard functions
- be able to use the product and quotient rule for finding derivatives

Learning Outcomes

On completion you should be able to ...

- differentiate a function of a function using the chain rule
- differentiate a power function

1. The meaning of a function of a function

When we use a function like $\sin 2x$ or $e^{\ln x}$ or $\sqrt{x^2+1}$ we are in fact dealing with a composite function or **function of a function**.

$\sin 2x$ is the sine function of $2x$. This is, in fact, how we 'read' it:

$\sin 2x$ is read 'sine of $2x$'

Similarly $e^{\ln x}$ is the exponential function of the logarithm of x:

$e^{\ln x}$ is read 'e to the power of $\ln x$'

Finally $\sqrt{x^2+1}$ is also a composite function. It is the square root function of the polynomial x^2+1:

$\sqrt{x^2+1}$ is read as the 'square root of (x^2+1)'

When we talk about a function of a function in a general setting we will use the notation $f(g(x))$ where both f and g are functions.

Example 11

Specify the functions $f,\ g$ for the composite functions

(a) $\sin 2x$ (b) $\sqrt{x^2+1}$ (c) $e^{\ln x}$

Solution

(a) Here f is the sine function and g is the polynomial $2x$. We often write:

$f(g) = \sin g$ and $g(x) = 2x$

(b) Here $f(g) = \sqrt{g}$ and $g(x) = x^2 + 1$

(c) Here $f(g) = e^g$ and $g(x) = \ln x$

In each case the original function of x is obtained when $g(x)$ is substituted into $f(g)$.

Task

Specify the functions $f, \ g$ for the composite functions
(a) $\cos(3x^2-1)$ (b) $\sinh(e^x)$ (c) $(x^2+3x-1)^{1/3}$

Your solution
(a)

Answer
$f(g) = \cos g \qquad g(x) = 3x^2 - 1$

Your solution
(b)

Answer
$f(g) = \sinh g \qquad g(x) = e^x$

Your solution
(c)

Answer
$f(g) = g^{1/3} \qquad g(x) = x^2 + 3x - 1$

2. The derivative of a function of a function

To differentiate a function of a function we use the following Key Point:

Key Point 11

The Chain Rule

If $y = f(g(x))$, that is, a function of a function, then

$$\frac{dy}{dx} = \frac{df}{dg} \times \frac{dg}{dx}$$

This is called the **chain rule**.

Example 12

Find the derivatives of the following composite functions using the chain rule and check the result using other methods

(a) $(2x^2-1)^2$ (b) $\ln e^x$

Solution

(a) Here $y = f(g(x))$ where $f(g) = g^2$ and $g(x) = 2x^2 - 1$. Thus

$$\frac{df}{dg} = 2g \quad \text{and} \quad \frac{dg}{dx} = 4x \quad \therefore \quad \frac{dy}{dx} = 2g.(4x) = 2(2x^2-1)(4x) = 8x(2x^2-1)$$

This result is easily checked by using the rule for differentiating products:

$y = (2x^2-1)(2x^2-1)$ so $\dfrac{dy}{dx} = 4x(2x^2-1)+(2x^2-1)(4x) = 8x(2x^2-1)$ as obtained above.

(b) Here $y = f(g(x))$ where $f(g) = \ln g$ and $g(x) = e^x$. Thus

$$\frac{df}{dg} = \frac{1}{g} \quad \text{and} \quad \frac{dg}{dx} = e^x \quad \therefore \quad \frac{dy}{dx} = \frac{1}{g}\cdot e^x = \frac{1}{e^x}\cdot e^x = 1$$

This is easily checked since, of course,

$y = \ln e^x = x$ and so, obviously $\dfrac{dy}{dx} = 1$ as obtained above.

Task

Obtain the derivatives of the following functions

(a) $(2x^2-5x+3)^9$ (b) $\sin(\cos x)$ (c) $\left(\dfrac{2x+1}{2x-1}\right)^3$

(a) Specify f and g for the first function:

Your solution

$f(g) =$ $g(x) =$

Answer

$f(g) = g^9$ $g(x) = 2x^2 - 5x + 3$

Now obtain the derivative using the chain rule:

Your solution

Answer

$9(2x^2-5x+3)^8(4x-5)$. Can you see how to obtain the derivative without going through the intermediate stage of specifying f, g?

(b) Specify f and g for the second function:

Your solution

Answer

$f(g) = \sin g \qquad g(x) = \cos x$

Now use the chain rule to obtain the derivative:

Your solution

Answer

$-[\cos(\cos x)]\sin x$

(c) Apply the chain rule to the third function:

Your solution

Answer

$-\dfrac{12(2x+1)^2}{(2x-1)^4}$

3. Power functions

An example of a function of a function which often occurs is the so-called power function $[g(x)]^k$ where k is any rational number. This is an example of a function of a function in which

$$f(g) = g^k$$

Thus, using the chain rule: if $\quad y = [g(x)]^k \quad$ then $\quad \dfrac{dy}{dx} = \dfrac{df}{dg} \cdot \dfrac{dg}{dx} = k\, g^{k-1}\dfrac{dg}{dx}.$

For example, if $y = (\sin x + \cos x)^{1/3}$ then $\dfrac{dy}{dx} = \dfrac{1}{3}(\sin x + \cos x)^{-2/3}(\cos x - \sin x)$.

Task

Find the derivatives of the following power functions

(a) $y = \sin^3 x$ (b) $y = (x^2+1)^{1/2}$ (c) $y = (e^{3x})^7$

(a) Note that $\sin^3 x$ is the conventional way of writing $(\sin x)^3$. Now find its derivative:

Your solution

Answer

$\dfrac{dy}{dx} = 3(\sin x)^2 \cos x$ which we would normally write as $3\sin^2 x \cos x$

(b) Use the function of a function approach again:

Your solution

Answer

$\dfrac{dy}{dx} = \dfrac{1}{2}(x^2+1)^{-1/2}2x = \dfrac{x}{\sqrt{x^2+1}}$

(c) Use the function of a function approach first, and then look for a quicker way in this case:

Your solution

Answer

$\dfrac{dy}{dx} = 7(e^{3x})^6(3e^{3x}) = 21(e^{3x})^7 = 21e^{21x}$

Note that $(e^{3x})^7 = e^{21x}$ $\quad\therefore\quad$ $\dfrac{dy}{dx} = 21e^{21x}$ directly - a much quicker way.

Exercise

Obtain the derivatives of the following functions:

(a) $\left(\dfrac{2x+1}{3x-1}\right)^4$ (b) $\tan(3x^2+2x)$ (c) $\sin^2(3x^2-1)$

Answer

(a) $-\dfrac{20(2x+1)^3}{(3x-1)^5}$ (b) $2(3x+1)\sec^2(3x^2+2x)$

(c) $6x\sin(6x^2-2)$ (remember $\sin 2x \equiv 2\sin x\cos x$)