

# Using a Table of Derivatives

## 11.2

### Introduction

In Section 11.1 you were introduced to the idea of a derivative and you calculated some derivatives from first principles. Rather than calculating the derivative of a function from first principles it is common practice to use a **table of derivatives**. This Section provides such a table and shows you how to use it.

### Prerequisites

Before starting this Section you should ...

- understand the meaning of the term 'derivative'
- understand what is meant by the notation  $\frac{dy}{dx}$

### Learning Outcomes

On completion you should be able to ...

- use a table of derivatives to perform differentiation

# 1. Table of derivatives

Table 1 lists some of the common functions used in engineering and their corresponding derivatives. Remember that in each case the function in the right-hand column gives the rate of change, or the gradient of the graph, of the function on the left at a particular value of  $x$ .

N.B. The angle must always be in radians when differentiating trigonometric functions.

**Table 1**  
Common functions and their derivatives  
(In this table  $k$ ,  $n$  and  $c$  are constants)

Function	Derivative
constant	0
$x$	1
$kx$	$k$
$x^n$	$nx^{n-1}$
$kx^n$	$knx^{n-1}$
$e^x$	$e^x$
$e^{kx}$	$ke^{kx}$
$\ln x$	$1/x$
$\ln kx$	$1/x$
$\sin x$	$\cos x$
$\sin kx$	$k \cos kx$
$\sin(kx + c)$	$k \cos(kx + c)$
$\cos x$	$-\sin x$
$\cos kx$	$-k \sin kx$
$\cos(kx + c)$	$-k \sin(kx + c)$
$\tan x$	$\sec^2 x$
$\tan kx$	$k \sec^2 kx$
$\tan(kx + c)$	$k \sec^2(kx + c)$

In the trigonometric functions the angle is in radians.



## Key Point 4

Particularly important is the rule for differentiating powers of functions:

$$\text{If } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}$$

For example, if  $y = x^3$  then  $\frac{dy}{dx} = 3x^2$ .

**Example 2**

Use Table 1 to find  $\frac{dy}{dx}$  when  $y$  is given by (a)  $7x$  (b)  $14$  (c)  $5x^2$  (d)  $4x^7$

**Solution**

(a) We note that  $7x$  is of the form  $kx$  where  $k = 7$ . Using Table 1 we then have  $\frac{dy}{dx} = 7$ .

(b) Noting that  $14$  is a constant we see that  $\frac{dy}{dx} = 0$ .

(c) We see that  $5x^2$  is of the form  $kx^n$ , with  $k = 5$  and  $n = 2$ . The derivative,  $knx^{n-1}$ , is then  $10x^1$ , or more simply,  $10x$ . So if  $y = 5x^2$ , then  $\frac{dy}{dx} = 10x$ .

(d) We see that  $4x^7$  is of the form  $kx^n$ , with  $k = 4$  and  $n = 7$ . Hence the derivative,  $\frac{dy}{dx}$ , is given by  $28x^6$ .



Use Table 1 to find  $\frac{dy}{dx}$  when  $y$  is (a)  $\sqrt{x}$  (b)  $\frac{5}{x^3}$

(a) Write  $\sqrt{x}$  as  $x^{\frac{1}{2}}$ , and use the result for differentiating  $x^n$  with  $n = \frac{1}{2}$ .

**Your solution****Answer**

$$\frac{dy}{dx} = nx^{n-1} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}. \text{ This may be written as } \frac{1}{2\sqrt{x}}.$$

(b) Write  $\frac{5}{x^3}$  as  $5x^{-3}$  and use the result for differentiating  $kx^n$  with  $k = 5$  and  $n = -3$ .

**Your solution****Answer**

$$5(-3)x^{-3-1} = -15x^{-4}$$

Although Table 1 is written using  $x$  as the independent variable, the Table can be used for any variable.



Use Table 1 to find

- (a)  $\frac{dz}{dt}$  given  $z = e^t$  (b)  $\frac{dp}{dt}$  given  $p = e^{8t}$  (c)  $\frac{dz}{dy}$  given  $z = e^{-3y}$

(a)

**Your solution**

$$\frac{dz}{dt} =$$

**Answer**

From Table 1, if  $y = e^x$ , then  $\frac{dy}{dx} = e^x$ . Hence if  $z = e^t$  then  $\frac{dz}{dt} = e^t$ .

(b)

**Your solution**

$$\frac{dp}{dt} =$$

**Answer**

$$8e^{8t}$$

(c)

**Your solution**

$$\frac{dz}{dy} =$$

**Answer**

$$-3e^{-3y}$$



Find the derivative,  $\frac{dy}{dx}$ , when  $y$  is (a)  $\sin 2x$  (b)  $\cos \frac{x}{2}$  (c)  $\tan 5x$

(a) Use the result for  $\sin kx$  in Table 1, taking  $k = 2$ :

**Your solution**

$$\frac{dy}{dx} =$$

**Answer**

$$2 \cos 2x$$

(b) Note that  $\cos \frac{x}{2}$  is the same as  $\cos \frac{1}{2}x$ . Use the result for  $\cos kx$  in Table 1:

**Your solution**

$$\frac{dy}{dx} =$$

**Answer**

$$-\frac{1}{2} \sin \frac{x}{2}$$

(c) Use the result for  $\tan kx$  in Table 1:

**Your solution**

$$\frac{dy}{dx} =$$

**Answer**

$$5 \sec^2 5x$$

## Exercises

1. Find the derivatives of the following functions with respect to  $x$ :

(a)  $9x^2$  (b)  $5$  (c)  $6x^3$  (d)  $-13x^4$

2. Find  $\frac{dz}{dt}$  when  $z$  is given by:

(a)  $\frac{5}{t^3}$  (b)  $\sqrt{t^3}$  (c)  $5t^{-2}$  (d)  $-\frac{3}{2}t^{\frac{3}{2}}$  (e)  $\ln 5t$

3. Find the derivative of each of the following with respect to the appropriate variable:

(a)  $\sin 5x$  (b)  $\cos 4t$  (c)  $\tan 3r$  (d)  $e^{2v}$  (e)  $\frac{1}{e^{3t}}$

4. Find the derivatives of the following with respect to  $x$ :

(a)  $\cos \frac{2x}{3}$  (b)  $\sin(-2x)$  (c)  $\tan \pi x$  (d)  $e^{\frac{x}{2}}$  (e)  $\ln \frac{2}{3}x$

**Answers**

1. (a)  $18x$  (b)  $0$  (c)  $18x^2$  (d)  $-52x^3$

2. (a)  $-15t^{-4}$  (b)  $\frac{3}{2}t^{\frac{1}{2}}$  (c)  $-10t^{-3}$  (d)  $-\frac{9}{4}t^{\frac{1}{2}}$  (e)  $\frac{1}{t}$

3. (a)  $5 \cos 5x$  (b)  $-4 \sin 4t$  (c)  $3 \sec^2 3r$  (d)  $2e^{2v}$  (e)  $-3e^{-3t}$

4. (a)  $-\frac{2}{3} \sin \frac{2x}{3}$  (b)  $-2 \cos(-2x)$  (c)  $\pi \sec^2 \pi x$  (d)  $\frac{1}{2}e^{\frac{x}{2}}$  (e)  $\frac{1}{x}$



## Engineering Example 1

### Electrostatic potential

#### Introduction

The electrostatic potential due to a point charge  $Q$  coulombs at a position  $r$  (m) from the charge is given by

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

where  $\epsilon_0$ , the permittivity of free space,  $\approx 8.85 \times 10^{-12}$  F m<sup>-1</sup> and  $\pi \approx 3.14$ .

The field strength at position  $r$  is given by  $E = -\frac{dV}{dr}$ .

#### Problem in words

Find the electric field strength at a distance of 5 m from a source with a charge of 1 coulomb.

#### Mathematical statement of the problem

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

Substitute for  $\epsilon_0$  and  $\pi$  and use  $Q = 1$ , so that

$$V = \frac{1}{4 \times 3.14 \times 8.85 \times 10^{-12} r} \approx \frac{9 \times 10^9}{r} = 9 \times 10^9 r^{-1}$$

We need to differentiate  $V$  in order to find the electric field strength from the relationship  $E = -\frac{dV}{dr}$

#### Mathematical analysis

$$E = -\frac{dV}{dr} = -9 \times 10^9 (-r^{-2}) = 9 \times 10^9 r^{-2}$$

When  $r = 5$ ,

$$E = \frac{9 \times 10^9}{25} = 3.6 \times 10^8 \text{ (V m}^{-1}\text{.)}$$

#### Interpretation

The electric field strength is  $3.6 \times 10^8$  V m<sup>-1</sup> at  $r = 5$  m.

Note that the field potential varies with the reciprocal of distance (i.e. inverse linear law with distance) whereas the field strength obeys an inverse square law with distance.

## 2. Extending the table of derivatives

We now quote simple rules which enable us to extend the range of functions which we can differentiate. The first two rules are for differentiating **sums** or **differences** of functions. The reader should note that all of the rules quoted below can be obtained from first principles using the approach outlined in Section 11.1.



### Key Point 5

Rule 1: The derivative of  $f(x) + g(x)$  is  $\frac{df}{dx} + \frac{dg}{dx}$

Rule 2: The derivative of  $f(x) - g(x)$  is  $\frac{df}{dx} - \frac{dg}{dx}$

These rules say that to find the derivative of the sum (or difference) of two functions, we simply calculate the sum (or difference) of the derivatives of each function.



### Example 3

Find the derivative of  $y = x^6 + x^4$ .

#### Solution

We simply calculate the sum of the derivatives of each separate function:

$$\frac{dy}{dx} = 6x^5 + 4x^3$$

The third rule tells us how to differentiate a **multiple** of a function. We have already met and applied particular cases of this rule which appear in Table 1.



### Key Point 6

Rule 3: If  $k$  is a constant, the derivative of  $kf(x)$  is  $k\frac{df}{dx}$

This rule tells us that if a function is multiplied by a constant,  $k$ , then the derivative is also multiplied by the same constant,  $k$ .



### Example 4

Find the derivative of  $y = 8e^{2x}$

#### Solution

Here we are interested in differentiating a multiple of the function  $e^{2x}$ . We differentiate  $e^{2x}$ , giving  $2e^{2x}$ , and multiply the result by 8. Thus

$$\frac{dy}{dx} = 8 \times 2e^{2x} = 16e^{2x}$$



### Example 5

Find the derivative of  $y = 6 \sin 2x + 3x^2 - 5e^{3x}$

#### Solution

We differentiate each part of the function in turn.

$$\begin{aligned} y &= 6 \sin 2x + 3x^2 - 5e^{3x} \\ \frac{dy}{dx} &= 6(2 \cos 2x) + 3(2x) - 5(3e^{3x}) \\ &= 12 \cos 2x + 6x - 15e^{3x} \end{aligned}$$



Find  $\frac{dy}{dx}$  where  $y = 7x^5 - 3e^{5x}$ .

First find the derivative of  $7x^5$ :

**Your solution**

**Answer**

$$7(5x^4) = 35x^4$$



Next find the derivative of  $3e^{5x}$ :

**Your solution**

**Answer**

$$3(5e^{5x}) = 15e^{5x}$$

Combine your results to find the derivative of  $7x^5 - 3e^{5x}$ :

**Your solution**

$$\frac{dy}{dx} =$$

**Answer**

$$35x^4 - 15e^{5x}$$



Find  $\frac{dy}{dx}$  where  $y = 4 \cos \frac{x}{2} + 17 - 9x^3$ .

First find the derivative of  $4 \cos \frac{x}{2}$ :

**Your solution**

**Answer**

$$4\left(-\frac{1}{2} \sin \frac{x}{2}\right) = -2 \sin \frac{x}{2}$$

Next find the derivative of 17:

**Your solution**

**Answer**

0

Then find the derivative of  $-9x^3$ :

**Your solution**

**Answer**

$$3(-9x^2) = -27x^2$$

Finally state the derivative of  $y = 4 \cos \frac{x}{2} + 17 - 9x^3$ :

**Your solution**

$$\frac{dy}{dx} =$$

**Answer**

$$-2 \sin \frac{x}{2} - 27x^2$$

### Exercises

1. Find  $\frac{dy}{dx}$  when  $y$  is given by:

(a)  $3x^7 + 8x^3$  (b)  $-3x^4 + 2x^{1.5}$  (c)  $\frac{9}{x^2} + \frac{14}{x} - 3x$  (d)  $\frac{3+2x}{4}$  (e)  $(2+3x)^2$

2. Find the derivative of each of the following functions:

(a)  $z(t) = 5 \sin t + \sin 5t$  (b)  $h(v) = 3 \cos 2v - 6 \sin \frac{v}{2}$   
 (c)  $m(n) = 4e^{2n} + \frac{2}{e^{2n}} + \frac{n^2}{2}$  (d)  $H(t) = \frac{e^{3t}}{2} + 2 \tan 2t$  (e)  $S(r) = (r^2 + 1)^2 - 4e^{-2r}$

3. Differentiate the following functions.

(a)  $A(t) = (3 + e^t)^2$  (b)  $B(s) = \pi e^{2s} + \frac{1}{s} + 2 \sin \pi s$   
 (c)  $V(r) = (1 + \frac{1}{r})^2 + (r + 1)^2$  (d)  $M(\theta) = 6 \sin 2\theta - 2 \cos \frac{\theta}{4} + 2\theta^2$   
 (e)  $H(t) = 4 \tan 3t + 3 \sin 2t - 2 \cos 4t$

**Answers**

1. (a)  $21x^6 + 24x^2$  (b)  $-12x^3 + 3x^{0.5}$  (c)  $-\frac{18}{x^3} - \frac{14}{x^2} - 3$  (d)  $\frac{1}{2}$  (e)  $12 + 18x$   
 2. (a)  $z' = 5 \cos t + 5 \cos 5t$  (b)  $h' = -6 \sin 2v - 3 \cos \frac{v}{2}$  (c)  $m' = 8e^{2n} - 4e^{-2n} + n$   
 (d)  $H' = \frac{3e^{3t}}{2} + 4 \sec^2 2t$  (e)  $S' = 4r^3 + 4r + 8e^{-2r}$   
 3. (a)  $A' = 6e^t + 2e^{2t}$  (b)  $B' = 2\pi e^{2s} - \frac{1}{s^2} + 2\pi \cos(\pi s)$   
 (c)  $V' = -\frac{2}{r^2} - \frac{2}{r^3} + 2r + 2$  (d)  $M' = 12 \cos 2\theta + \frac{1}{2} \sin \frac{\theta}{4} + 4\theta$   
 (e)  $H' = 12 \sec^2 3t + 6 \cos 2t + 8 \sin 4t$

### 3. Evaluating a derivative

The need to find the rate of change of a function at a particular point occurs often. We do this by finding the derivative of the function, and then evaluating the derivative at that point. When taking derivatives of trigonometric functions, any angles **must** be measured in radians. Consider a function,  $y(x)$ . We use the notation  $\frac{dy}{dx}(a)$  or  $y'(a)$  to denote the derivative of  $y$  evaluated at  $x = a$ . So  $y'(0.5)$  means the value of the derivative of  $y$  when  $x = 0.5$ .



#### Example 6

Find the value of the derivative of  $y = x^3$  where  $x = 2$ . Interpret your result.

#### Solution

We have  $y = x^3$  and so  $\frac{dy}{dx} = 3x^2$ .

When  $x = 2$ ,  $\frac{dy}{dx} = 3(2)^2 = 12$ , that is,  $\frac{dy}{dx}(2) = 12$  (Equivalently,  $y'(2) = 12$ ).

The derivative is positive when  $x = 2$  and so  $y$  is increasing at this point. When  $x = 2$ ,  $y$  is increasing at a rate of 12 vertical units per horizontal unit.



#### Engineering Example 2

### Electromotive force

#### Introduction

Potential difference in an electrical circuit is produced by **electromotive force** (e.m.f.) which is measured in volts and describes the force that maintains current flow around a closed path. Every source of continuous electrical energy, including batteries, generators and thermocouples, consist essentially of an energy converter that produces an e.m.f. An electric current always produces a magnetic field. So the current  $i$  which flows round any closed path produces a magnetic flux  $\phi$  which passes through that path. Conversely, if another closed path, i.e. another coil, is placed within the first path, then the magnetic field due to the first circuit can induce an e.m.f. and hence a current in the second coil. The simplest closed path is a single loop. More commonly, helical coils, known as search coils, with known area and number of turns, are used. The induced e.m.f. depends upon the number of turns in the coil. The search coil is used with a fluxmeter to measure the change of flux linkage.

#### Problem in words

A current  $i$  is travelling through a single turn loop of radius 1 m. A 4-turn search coil of effective area  $0.03 \text{ m}^2$  is placed inside the loop. The magnetic flux in weber (Wb) linking the search coil is given by:

$$\phi = \mu_0 \frac{iA}{2r}$$

where  $r$  (m) is the radius of the current carrying loop,  $A$  ( $\text{m}^2$ ) is the area of the search coil and  $\mu_0$  is the permeability of free space,  $4 \times 10^{-7} \text{ H m}^{-1}$ .

Find the e.m.f. (in volts) induced in the search coil, given by  $\varepsilon = -N \left( \frac{d\phi}{dt} \right)$  where  $N$  is the number of turns in the search coil, and the current is given by

$$i = 20 \sin(20\pi t) + 50 \sin(30\pi t)$$

### Mathematical statement of the problem

Substitute  $i = 20 \sin(20\pi t) + 50 \sin(30\pi t)$  into  $\phi = \mu_0 \frac{iA}{2r}$  and find  $\varepsilon = -N \left( \frac{d\phi}{dt} \right)$  when  $r = 1 \text{ m}$ ,  $A = 0.03 \text{ m}^2$ ,  $\mu_0 = 4 \times 10^{-7} \text{ H m}^{-1}$  and  $N = 4$ .

### Mathematical analysis

$$\phi = \mu_0 \frac{iA}{2r} = \frac{\mu_0 A}{2r} (20 \sin(20\pi t) + 50 \sin(30\pi t))$$

$$\begin{aligned} \text{So } \frac{d\phi}{dt} &= \frac{\mu_0 A}{2r} (20\pi \times 20 \cos(20\pi t) + 30\pi \times 50 \cos(30\pi t)) \\ &= \frac{\mu_0 A}{2r} (400\pi \cos(20\pi t) + 1500\pi \cos(30\pi t)) \end{aligned}$$

$$\text{so } \varepsilon = -N \frac{d\phi}{dt} = -4 \frac{\mu_0 A}{2r} (400\pi \cos(20\pi t) + 1500\pi \cos(30\pi t))$$

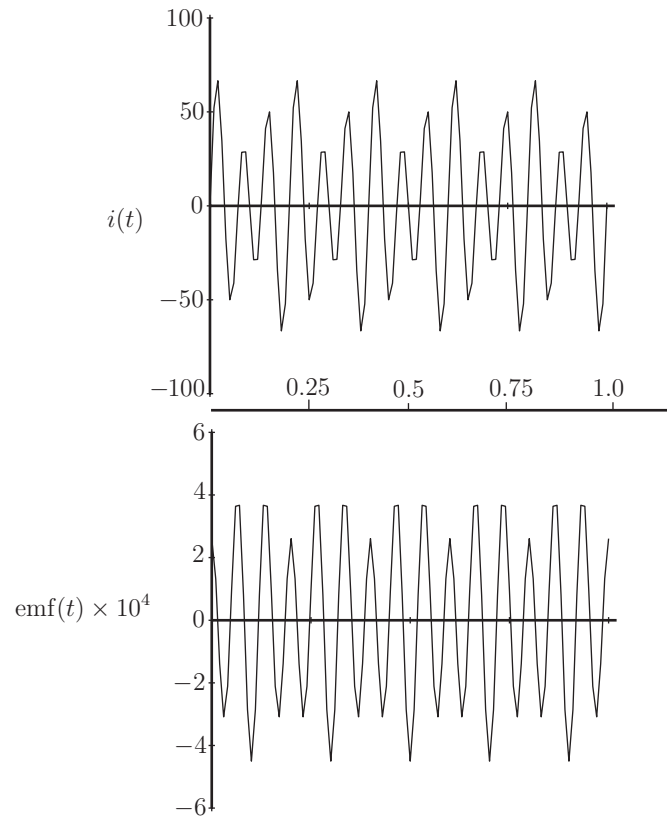
Now,  $\mu_0 = 4\pi \times 10^{-7}$ ,  $A = 0.03$  and  $r = 1$

$$\begin{aligned} \text{So } \varepsilon &= \frac{-4 \times 4\pi \times 10^{-7} \times 0.03}{2 \times 1} (400\pi \cos(20\pi t) + 1500\pi \cos(30\pi t)) \\ &= -7.5398 \times 10^{-8} (1256.64 \cos(20\pi t) + 4712.39 \cos(30\pi t)) \\ &= -9.475 \times 10^{-5} \cos(20\pi t) + 3.553 \times 10^{-4} \cos(30\pi t) \end{aligned}$$

### Interpretation

The induced e.m.f. is  $-9.475 \times 10^{-5} \cos(20\pi t) + 3.553 \times 10^{-4} \cos(30\pi t)$ .

The graphs in Figure 8 show the initial current in the single loop and the e.m.f. induced in the search coil.



**Figure 8**

Note that the induced e.m.f. does not start at zero, which the initial current does, and has a different pattern of variation with time.

### Exercises

1. Calculate the derivative of  $y = x^2 + \sin x$  when  $x = 0.2$  radians.
2. Calculate the rate of change of  $i(t) = 4 \sin 2t + 3t$  when
  - (a)  $t = \frac{\pi}{3}$  (b)  $t = 0.6$  radians
3. Calculate the rate of change of  $F(t) = 5 \sin t - 3 \cos 2t$  when
  - (a)  $t = 0$  radians (b)  $t = 1.3$  radians

#### Answers

1. 1.380
2. (a)  $-1$  (b) 5.8989
3. (a) 5 (b) 4.4305