The Binomial Series 16.3

Introduction

In this Section we examine an important example of an infinite series, the **binomial** series:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

We show that this series is only convergent if $|x| < 1$ and that in this case the series sums to the value $(1+x)^p$. As a special case of the binomial series we consider the situation when p is a positive integer n. In this case the infinite series reduces to a **finite** series and we obtain, by replacing x with $\frac{b}{a}$, the **binomial theorem**:

$$(b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{2!}b^{n-2}a^2 + \cdots + a^n.$$

Finally, we use the binomial series to obtain various polynomial expressions for $(1+x)^p$ when x is 'small'.

Prerequisites

Before starting this Section you should ...

- understand the factorial notation
- have knowledge of the ratio test for convergence of infinite series.
- understand the use of inequalities

Learning Outcomes

On completion you should be able to ...

- recognise and use the binomial series
- state and use the binomial theorem
- use the binomial series to obtain numerical approximations

1. The binomial series

A very important infinite series which occurs often in applications and in algebra has the form:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

in which p is a given number and x is a variable. By using the ratio test it can be shown that this series converges, irrespective of the value of p, as long as $|x| < 1$. In fact, as we shall see in Section 16.5 the given series converges to the value $(1+x)^p$ as long as $|x| < 1$.

Key Point 9

The Binomial Series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \qquad |x| < 1$$

The binomial theorem can be obtained directly from the binomial series if p is chosen to be a **positive integer** (here we need not demand that $|x| < 1$ as the series is now finite and so is always convergent irrespective of the value of x). For example, with $p = 2$ we obtain

$$\begin{aligned}(1+x)^2 &= 1 + 2x + \frac{2(1)}{2}x^2 + 0 + 0 + \cdots \\ &= 1 + 2x + x^2 \quad \text{as is well known.}\end{aligned}$$

With $p = 3$ we get

$$\begin{aligned}(1+x)^3 &= 1 + 3x + \frac{3(2)}{2}x^2 + \frac{3(2)(1)}{3!}x^3 + 0 + 0 + \cdots \\ &= 1 + 3x + 3x^2 + x^3\end{aligned}$$

Generally if $p = n$ (a positive integer) then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

which is a form of the binomial theorem. If x is replaced by $\dfrac{b}{a}$ then

$$\left(1 + \frac{b}{a}\right)^n = 1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \cdots + \left(\frac{b}{a}\right)^n$$

Now multiplying both sides by a^n we have the following Key Point:

Key Point 10

The Binomial Theorem

If n is a positive integer then the expansion of $(a+b)$ raised to the power n is given by:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + b^n$$

This is known as the **binomial** theorem.

Task

Use the binomial theorem to obtain (a) $(1+x)^7$ (b) $(a+b)^4$

(a) Here $n=7$:

Your solution

$(1+x)^7 =$

Answer

$(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

(b) Here $n=4$:

Your solution

$(a+b)^4 =$

Answer

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$

Task

Given that x is so small that powers of x^3 and above may be ignored in comparison to lower order terms, find a quadratic approximation of $(1-x)^{\frac{1}{2}}$ and check for accuracy your approximation for $x = 0.1$.

First expand $(1-x)^{\frac{1}{2}}$ using the binomial series with $p = \dfrac{1}{2}$ and with x replaced by $(-x)$:

Your solution

$(1-x)^{\frac{1}{2}} =$

Answer

$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2}x^2 - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^3 + \cdots$

Now obtain the quadratic approximation:

Your solution

$$(1-x)^{\frac{1}{2}} \simeq$$

Answer

$(1-x)^{\frac{1}{2}} \simeq 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2$

Now check on the validity of the approximation by choosing $x = 0.1$:

Your solution

Answer

On the left-hand side we have

$(0.9)^{\frac{1}{2}} = 0.94868$ to 5 d.p. obtained by calculator

whereas, using the quadratic expansion:

$$(0.9)^{\frac{1}{2}} \approx 1 - \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 = 1 - 0.05 - (0.00125) = 0.94875.$$

so the error is only 0.00007.

What we have done in this last Task is to replace (or approximate) the function $(1-x)^{\frac{1}{2}}$ by the simpler (polynomial) function $1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2$ which is reasonable provided x is very small. This approximation is well illustrated geometrically by drawing the curves $y = (1-x)^{\frac{1}{2}}$ and $y = 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2$. The two curves coincide when x is 'small'. See Figure 2:

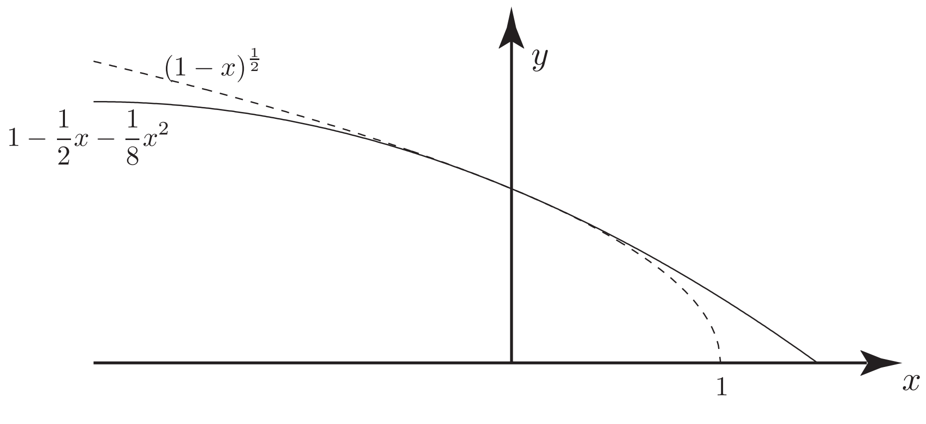

Figure 2

Task

Obtain a cubic approximation of $\dfrac{1}{(2+x)}$. Check your approximation for accuracy using appropriate values of x.

First write the term $\dfrac{1}{(2+x)}$ in a form suitable for the binomial series (refer to Key Point 9):

Your solution

$$\frac{1}{(2+x)} =$$

Answer

$$\frac{1}{2+x} = \frac{1}{2\left(1+\dfrac{x}{2}\right)} = \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$

Now expand using the binomial series with $p = -1$ and $\dfrac{x}{2}$ instead of x, to include terms up to x^3:

Your solution

$$\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} =$$

Answer

$$\begin{aligned}\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} &= \frac{1}{2}\left\{1+(-1)\frac{x}{2}+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3\right\}\\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}\end{aligned}$$

State the range of x for which the binomial series of $\left(1+\dfrac{x}{2}\right)^{-1}$ is valid:

Your solution

The series is valid if

Answer

valid as long as $\left|\dfrac{x}{2}\right| < 1$ i.e. $|x| < 2$ or $-2 < x < 2$

Choose $x = 0.1$ to check the accuracy of your approximation:

Your solution

$$\frac{1}{2}\left(1+\frac{0.1}{2}\right)^{-1} =$$

$$\frac{1}{2}-\frac{0.1}{4}+\frac{0.01}{8}-\frac{0.001}{16} =$$

Answer

$$\frac{1}{2}\left(1+\frac{0.1}{2}\right)^{-1} = 0.47619 \text{ to 5 d.p.}$$

$$\frac{1}{2}-\frac{0.1}{4}+\frac{0.01}{8}-\frac{0.001}{16} = 0.4761875.$$

Figure 3 below illustrates the close correspondence (when x is 'small') between the curves $y = \frac{1}{2}(1+\frac{x}{2})^{-1}$ and $y = \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}$.

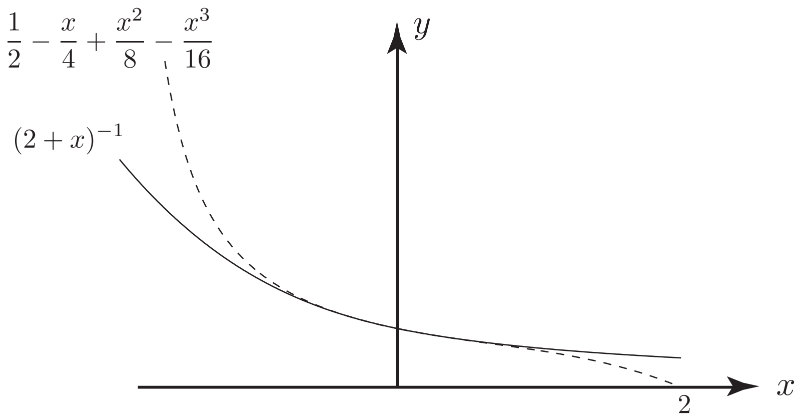

Figure 3

Exercises

1. Determine the expansion of each of the following

   (a) $(a+b)^3$, (b) $(1-x)^5$, (c) $(1+x^2)^{-1}$, (d) $(1-x)^{1/3}$.

2. Obtain a cubic approximation (valid if x is small) of the function $(1+2x)^{3/2}$.

Answers

1. (a) $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

   (b) $(1-x)^5 = 1-5x+10x^2-10x^3+5x^4-x^5$

   (c) $(1+x^2)^{-1} = 1-x^2+x^4-x^6+\cdots$

   (d) $(1-x)^{1/3} = 1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3+\cdots$

2. $(1+2x)^{3/2} = 1+3x+\frac{3}{2}x^2-\frac{1}{2}x^3+\cdots$