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Integration

Learning outcomes

In this Workbook you will learn about integration and about some of the common techniques employed to obtain integrals. You will learn that integration is the inverse operation to differentiation and will also appreciate the distinction between a definite and an indefinite integral. You will understand how a definite integral is related to the area under a curve. You will understand how to use the technique of integration by parts to obtain integrals involving the product of functions. You will also learn how to use partial fractions and trigonometric identities in integration.

Basic Concepts of Integration

13.1

Introduction

When a function $f(x)$ is known we can differentiate it to obtain its derivative $\frac{df}{dx}$. The reverse process is to obtain the function $f(x)$ from knowledge of its derivative. This process is called **integration**. Applications of integration are numerous and some of these will be explored in subsequent Sections. First, what is important is to practise basic techniques and learn a variety of methods for integrating functions.

Prerequisites

Before starting this Section you should ...

- thoroughly understand the various techniques of differentiation

Learning Outcomes

On completion you should be able to ...

- evaluate simple integrals by reversing the process of differentiation
- use a table of integrals
- explain the need for a constant of integration when finding indefinite integrals
- use the rules for finding integrals of sums of functions and constant multiples of functions

1. Integration as differentiation in reverse

Suppose we differentiate the function $y = x^2$. We obtain $\dfrac{dy}{dx} = 2x$. Integration reverses this process and we say that the integral of $2x$ is x^2. Pictorially we can regard this as shown in Figure 1:

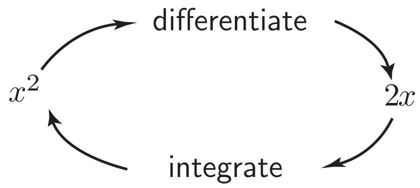

Figure 1

The situation is just a little more complicated because there are lots of functions we can differentiate to give $2x$. Here are some of them: $x^2 + 4$, $x^2 - 15$, $x^2 + 0.5$
All these functions have the same derivative, $2x$, because when we differentiate the constant term we obtain zero. Consequently, when we reverse the process, we have no idea what the original constant term might have been. So we include in our answer an unknown constant, c say, called the **constant of integration**. We state that the integral of $2x$ is $x^2 + c$.

When we want to differentiate a function, $y(x)$, we use the notation $\dfrac{d}{dx}$ as an instruction to differentiate, and write $\dfrac{d}{dx}\big(y(x)\big)$. In a similar way, when we want to integrate a function we use a special notation: $\displaystyle\int y(x)\,dx$.

The symbol for integration, $\displaystyle\int$, is known as an **integral sign**. To integrate $2x$ we write

$$\int \underbrace{2x}_{} \; \mathrm{d}x = x^2 + c$$

integral sign

this term is called the integrand

there must always be a term of the form $\mathrm{d}x$

constant of integration

Note that along with the integral sign there is a term of the form dx, which must always be written, and which indicates the variable involved, in this case x. We say that $2x$ is being **integrated with respect to x**. The function being integrated is called the **integrand**. Technically, integrals of this sort are called **indefinite integrals**, to distinguish them from definite integrals which are dealt with subsequently. When you find an indefinite integral your answer should always contain a constant of integration.

Exercises

1. (a) Write down the derivatives of each of: x^3, $x^3 + 17$, $x^3 - 21$

   (b) Deduce that $\displaystyle\int 3x^2\,dx = x^3 + c$.

2. Explain why, when finding an indefinite integral, a constant of integration is always needed.

Answers

1. (a) $3x^2$, $3x^2$, $3x^2$ (b) Whatever the constant, it is zero when differentiated.

2. Any constant will disappear (i.e. become zero) when differentiated so one must be reintroduced to reverse the
process.

2. A table of integrals

We could use a table of derivatives to find integrals, but the more common ones are usually found in a 'Table of Integrals' such as that shown below. You could check the entries in this table using your knowledge of differentiation. Try this for yourself.

Table 1: Integrals of Common Functions

function $f(x)$	indefinite integral $\int f(x)\,dx$
constant, k	$kx + c$
x	$\frac{1}{2}x^2 + c$
x^2	$\frac{1}{3}x^3 + c$
x^n	$\dfrac{x^{n+1}}{n+1} + c, \quad n \neq -1$
x^{-1} (or $\dfrac{1}{x}$)	$\ln\lvert x\rvert + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\cos kx$	$\dfrac{1}{k}\sin kx + c$
$\sin kx$	$-\dfrac{1}{k}\cos kx + c$
$\tan kx$	$\dfrac{1}{k}\ln\lvert\sec kx\rvert$+c
e^x	$\mathrm{e}^x + c$
e^{-x}	$-\mathrm{e}^{-x} + c$
e^{kx}	$\dfrac{1}{k}\mathrm{e}^{kx} + c$

When dealing with the trigonometric functions the variable x must always be measured in radians and not degrees. Note that the fourth entry in the Table, for x^n, is valid for any value of n, positive or negative, whole number or fractional, *except* $n = -1$. When $n = -1$ use the fifth entry in the Table.

Example 1

Use Table 1 to find the indefinite integral of x^7: that is, find $\int x^7\,dx$

Solution

From Table 1 note that $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$. In words, this states that to integrate a power of x, increase the power by 1, and then divide the result by the new power. With $n = 7$ we find

$$\int x^7\,dx = \frac{1}{8}x^8 + c$$

Example 2

Find the indefinite integral of $\cos 5x$: that is, find $\int \cos 5x\,dx$

Solution

From Table 1 note that $\int \cos kx\,dx = \dfrac{\sin kx}{k} + c$

With $k = 5$ we find $\int \cos 5x\,dx = \dfrac{1}{5}\sin 5x + c$

In Table 1 the independent variable is always given as x. However, with a little imagination you will be able to use it when other independent variables are involved.

Example 3

Find $\int \cos 5t\,dt$

Solution

We integrated $\cos 5x$ in the previous example. Now the independent variable is t, so simply use Table 1 and replace every x with a t. With $k = 5$ we find

$$\int \cos 5t\,dt = \frac{1}{5}\sin 5t + c$$

It follows immediately that, for example,

$$\int \cos 5\omega\,d\omega = \frac{1}{5}\sin 5\omega + c, \qquad \int \cos 5u\,du = \frac{1}{5}\sin 5u + c \quad \text{and so on.}$$

Example 4

Find the indefinite integral of $\dfrac{1}{x}$: that is, find $\displaystyle\int \frac{1}{x}\,dx$

Solution

This integral deserves special mention. You may be tempted to try to write the integrand as x^{-1} and use the fourth row of Table 1. However, the formula $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ is not valid when $n = -1$ as Table 1 makes clear. This is because we can never divide by zero. Look to the fifth entry of Table 1 and you will see $\displaystyle\int x^{-1}\,dx = \ln|x| + c$.

Example 5

Find $\displaystyle\int 12\,dx$ and $\displaystyle\int 12\,dt$

Solution

In this Example we are integrating a constant, 12. Using Table 1 we find

$\displaystyle\int 12\,dx = 12x + c$ Similarly $\displaystyle\int 12\,dt = 12t + c$.

Task

Find $\displaystyle\int t^4\,dt$

Your solution

Answer

$$\int t^4\,dt = \frac{1}{5}t^5 + c.$$

Task

Find $\displaystyle\int \frac{1}{x^5}\,dx$ using the laws of indices to write the integrand as x^{-5} and then use Table 1:

Your solution

Answer

$-\dfrac{1}{4}x^{-4} + c = -\dfrac{1}{4x^4} + c.$

Task

Find $\displaystyle\int \mathrm{e}^{-2x}\,dx$ using the entry in Table 1 for integrating e^{kx}:

Your solution

Answer

With $k = -2$, we have $\int \mathrm{e}^{-2x}\,dx = -\dfrac{1}{2}\mathrm{e}^{-2x} + c.$

Exercises

1. Integrate each of the following functions with respect to x:
   (a) x^9, (b) $x^{1/2}$, (c) x^{-3}, (d) $1/x^4$, (e) 4, (f) $\sqrt{x}$, (g) e^{4x}

2. Find (a) $\displaystyle\int t^2\,dt$, (b) $\displaystyle\int 6\,dt$, (c) $\displaystyle\int \sin 3t\,dt$, (d) $\displaystyle\int \mathrm{e}^{7t}\,dt$.

Answers

1 (a) $\dfrac{1}{10}x^{10} + c$, (b) $\dfrac{2}{3}x^{3/2} + c$, (c) $-\dfrac{1}{2}x^{-2} + c$, (d) $-\dfrac{1}{3}x^{-3} + c$, (e) $4x + c$,

(f) same as (b), (g) $\dfrac{1}{4}\mathrm{e}^{4x} + c$

2. (a) $\dfrac{1}{3}t^3 + c$, (b) $6t + c$, (c) $-\dfrac{1}{3}\cos 3t + c$, (d) $\dfrac{1}{7}\mathrm{e}^{7t} + c$

3. Some rules of integration

To enable us to find integrals of a wider range of functions than those normally given in a table of integrals we can make use of the following rules.

The integral of $k\,f(x)$ where k is a constant

A constant factor in an integral can be moved outside the integral sign as follows:

Key Point 1

$$\int k\,f(x)\,dx = k\int f(x)\,dx$$

Example 6

Find the indefinite integral of $11x^2$: that is, find $\displaystyle\int 11x^2\,dx$

Solution

$$\int 11x^2\,dx = 11\int x^2\,dx = 11\left(\frac{x^3}{3}+c\right) = \frac{11x^3}{3}+K \quad \text{where } K \text{ is a constant.}$$

Example 7

Find the indefinite integral of $-5\cos x$; that is, find $\displaystyle\int -5\cos x\,dx$

Solution

$$\int -5\cos x\,dx = -5\int \cos x\,dx = -5\,(\sin x + c) = -5\sin x + K \quad \text{where } K \text{ is a constant.}$$

The integral of $f(x)+g(x)$ and of $f(x)-g(x)$

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

Key Point 2

$$\int [\,f(x)+g(x)\,]\,dx = \int f(x)\,dx + \int g(x)\,dx$$
$$\int [\,f(x)-g(x)\,]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

Example 8

Find $\displaystyle\int (x^3+\sin x)\,dx$

Solution

$$\int (x^3+\sin x)\,dx = \int x^3\,dx + \int \sin x\,dx = \frac{1}{4}x^4 - \cos x + c$$

Note that only a single constant of integration is needed.

Task

Find $\displaystyle\int (3t^4+\sqrt{t})\,dt$

Use Key Points 1 and 2:

Your solution

Answer

$\dfrac{3}{5}t^5 + \dfrac{2}{3}t^{3/2} + c$

Task

The hyperbolic sine and cosine functions, $\sinh x$ and $\cosh x$, are defined as follows:

$$\sinh x = \frac{\mathsf{e}^x - \mathsf{e}^{-x}}{2} \qquad \cosh x = \frac{\mathsf{e}^x + \mathsf{e}^{-x}}{2}$$

Note that they are combinations of the exponential functions e^x and e^{-x}.
Find the indefinite integrals of $\sinh x$ and $\cosh x$.

Your solution

$$\int \sinh x\,dx = \int \left(\frac{\mathsf{e}^x - \mathsf{e}^{-x}}{2}\right)\,dx =$$

$$\int \cosh x\,dx = \int \left(\frac{\mathsf{e}^x + \mathsf{e}^{-x}}{2}\right)\,dx =$$

Answer

$$\int \sinh x\,dx = \frac{1}{2}\int \mathsf{e}^x\,dx - \frac{1}{2}\int \mathsf{e}^{-x}\,dx = \frac{1}{2}\mathsf{e}^x + \frac{1}{2}\mathsf{e}^{-x} + c = \frac{1}{2}\big(\mathsf{e}^x + \mathsf{e}^{-x}\big) + c = \cosh x + c.$$

Similarly $\displaystyle\int \cosh x\,dx = \sinh x + c$.

Further rules for finding more complicated integrals are dealt with in subsequent Sections.

Engineering Example 1

Electrostatic charge

Introduction

Electrostatic charge is important both where it is wanted, as in the electrostatic precipitator plate systems used for cleaning gases, and where it is unwanted, such as when charge builds up on moving belts. This Example is concerned with a charged object with a particular idealised shape - a sphere. However, similar analytical calculations can be carried out for certain other shapes and numerical methods can be used for more complicated shapes.
The electric field at all points inside and outside a charged sphere is given by

$$E(r) = \frac{Qr}{4\pi\varepsilon_0 a^3} \quad \text{if} \quad r < a \qquad (1a)$$

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{if} \quad r \geq a \qquad (1b)$$

where ε_0 is the permittivity of free space, Q is the total charge, a is the radius of the sphere, and r is the radial distance between the centre of the sphere and a point of observation (see Figure 2).

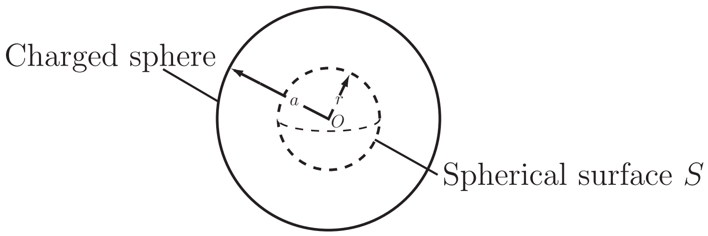

Figure 2: Geometry and symbols associated with the charged sphere

The electric field associated with electrostatic charge has a scalar potential. The electric field defined by (1a) and (1b) shows only a radial dependence of position. Therefore, the electric scalar potential $V(r)$ is related to the field $E(r)$ by

$$E(r) = -\frac{dV}{dr}. \qquad (2)$$

Problem in words

A sphere is charged with a uniform density of charge and no other charge is present outside the sphere in space. Determine the variation of electric potential with distance from the centre of the sphere.

Mathematical statement of problem

Determine the electric scalar potential as a function of r, $V(r)$, by integrating (2).

Mathematical analysis

Equation (2) yields $V(r)$ as the negative of the indefinite integral of $E(r)$.

$$-\int dV = \int E(r)\,dr. \tag{3}$$

Using (1a) and (1b) with (3) leads to

$$V(r) = -\frac{Q}{4\pi\varepsilon_0 a^3}\int r\,dr \quad \text{if} \quad r < a \tag{4a}$$

$$V(r) = -\frac{Q}{4\pi\varepsilon_0}\int \frac{dr}{r^2} \quad \text{if} \quad r \geq a \tag{4b}$$

Using the facts that $\int r\,dr = r^2/2 + c_1$ and $\int \frac{dr}{r^2} = -\frac{1}{r} + c_2$,
(4a) and (4b) become

$$V(r) = -\frac{Qr^2}{8\pi\varepsilon_0 a^3} + c_1 \quad \text{if} \quad r < a \tag{5a}$$

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} + c_2 \quad \text{if} \quad r \geq a \tag{5b}$$

The integration constant c_2 can be determined by assuming that the electric potential is zero at an infinite distance from the sphere:

$$\lim_{r\to\infty}[V(r)] = 0 \qquad \Rightarrow \qquad \lim_{r\to\infty}\left[-\frac{Q}{4\pi\varepsilon_0 r}\right] + c_2 = 0 \qquad \Rightarrow \qquad c_2 = 0.$$

The constant c_1 can be determined by assuming that the potential is continuous at $r = a$.
From equation (5a)

$$V(a) = -\frac{Qa^2}{8\pi\varepsilon_0 a^3} + c_1$$

From equation (5b)

$$V(a) = \frac{Q}{4\pi\varepsilon_0 a}$$

Hence

$$c_1 = \frac{Q}{4\pi\varepsilon_0 a} + \frac{2Q}{8\pi\varepsilon_0 a} = \frac{3Q}{8\pi\varepsilon_0 a}.$$

Substituting for c_1 in (5), the electric potential is obtained for all space is:

$$V(r) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3a^2 - r^2}{2a^3}\right) \quad \text{if} \quad r < a.$$

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} \quad \text{if} \quad r \geq a$$

Interpretation

The potential of the electrostatic field outside a charged sphere varies inversely with distance from the centre of the sphere. Inside the sphere, the electrostatic potential varies with the square of the distance from the centre.

An Engineering Exercise in HELM 29.3 derives the corresponding expressions for the variation of the electrostatic field and an Engineering Exercise in HELM 27.4 calculates the potential energy due to the charged sphere.

Exercises

1. Find $\int (2x - \mathsf{e}^x)\, dx$
2. Find $\int 3\mathsf{e}^{2x}\, dx$
3. Find $\int \frac{1}{3}(x + \cos 2x)\, dx$
4. Find $\int 7x^{-2}\, dx$
5. Find $\int (x+3)^2\, dx$, (be careful!)

Answers

1. $x^2 - \mathsf{e}^x + c$
2. $\frac{3}{2}\mathsf{e}^{2x} + c$
3. $\frac{1}{6}x^2 + \frac{1}{6}\sin 2x + c$
4. $-\frac{7}{x} + c$
5. $\frac{1}{3}x^3 + 3x^2 + 9x + c$