Completing the square maxima and minima

Completing the square is an algebraic technique which has several applications. These include the solution of quadratic equations. In this unit we use it to find the maximum or minimum values of quadratic functions.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature. To help you to achieve this, the unit includes a substantial number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- complete the square for a quadratic expression
- find maximum or minimum values of a quadratic function by completing the square

Contents

1. Introduction

Completing the Square is a technique which can be used to find maximum or minimum values of quadratic functions. We can also use this technique to change or simplify the form of algebraic expressions. We can use it for solving quadratic equations.

In this unit we will be using Completing the Square to find maximum and minimum values of quadratic functions.

2. The minimum value of a quadratic function

Consider the function

$$y = x^2 + 5x - 2$$

You may be aware from previous work that the graph of a quadratic function, where the coefficient of x^2 is positive as it is here, will take the form of one of the graphs shown in Figure 1.

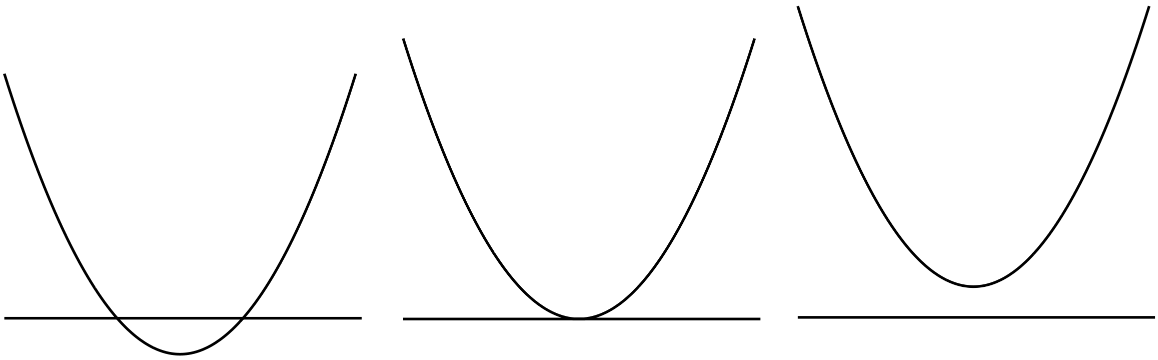

Figure 1. We may be interested in finding the coordinates of the minimum point

There will be a minimum, or lowest point on the graph and we may be interested in finding the x and y values at this point. Is the minimum point below or above the horizontal axis ? This question can be answered using techniques in calculus, but here as an alternative we use Completing the Square.

3. What is meant by a complete or exact square ?

An expression of the form

$$(x + a)^2$$

is called a **complete** or **exact** square. Multiplying this out we obtain

$$\begin{aligned} (x + a)^2 &= (x + a)(x + a) \\ &= x^2 + 2ax + a^2 \end{aligned}$$

In the same way, consider

$$(x - a)^2$$

Multiplying out the brackets:

$$\begin{aligned} (x - a)^2 &= (x - a)(x - a) \\ &= x^2 - 2ax + a^2 \end{aligned}$$

Both expressions $x^2 + 2ax + a^2$ and $x^2 - 2ax + a^2$ are called complete squares because they can be written as a single term squared, that is $(x + a)^2$, or $(x - a)^2$.

4. Completing the square - when the coefficient of x^2 is 1

We now return to the quadratic expression $x^2 + 5x - 2$ and we are going to try to write it in the form of a single term squared, that is a complete square, in this case $(x + a)^2$.

Compare the two expressions:

$$x^2 + 2ax + a^2 \qquad \text{and} \qquad x^2 + 5x - 2$$

Clearly the coefficients of x^2 in both expressions are the same - they match up.

We would like to match up the term $2ax$ with the term $5x$. To do this note that $2a$ must equal 5, so that $a = \frac{5}{2}$.
Recall that

$$(x + a)^2 = x^2 + 2ax + a^2$$

With this value of a

$$\left(x + \frac{5}{2}\right)^2 = x^2 + 5x + \left(\frac{5}{2}\right)^2$$

However, the right hand side is not the same as the original expression $x^2 + 5x - 2$. To make it the same we need to subtract $\left(\frac{5}{2}\right)^2$ to remove this unwanted term:

$$\left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 = x^2 + 5x + \cancel{\left(\frac{5}{2}\right)^2} - \cancel{\left(\frac{5}{2}\right)^2}$$

and then subtract 2 to insert the term we need:

$$\left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 2 = x^2 + 5x - 2$$

So we have

$$x^2 + 5x - 2 = \left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 2$$

Combining the last two numbers

$$x^2 + 5x - 2 = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$$

At this stage we have finished completing the square for the quadratic expression $x^2 + 5x - 2$. This expression is not a complete or exact square because in addition to the complete square $\left(x + \frac{5}{2}\right)^2$ there is the constant term $-\frac{33}{4}$.

Having written the expression $x^2 + 5x - 2$ in this form it is now straightforward to find its minimum value. This is because the first part, $\left(x + \frac{5}{2}\right)^2$ being a square, is always positive unless it is equal to zero. Zero is its lowest possible value and so the lowest possible value of $x^2 + 5x - 2$ must be $-\frac{33}{4}$. This lowest value will occur when $\left(x + \frac{5}{2}\right)^2$ is zero, that is when $x = -\frac{5}{2}$.

In conclusion, we have shown that the minimum value of $x^2 + 5x - 2$ is $-\frac{33}{4}$ and this occurs when $x = -\frac{5}{2}$.

Example

Suppose we wish to find the minimum value of the quadratic function $f(x) = x^2 - 6x - 12$.

Compare the two expressions:

$$x^2 - 2ax + a^2 \qquad \text{and} \qquad x^2 - 6x - 12$$

Clearly the coefficients of x^2 in both expressions are the same - they match up.

We would like to match up the term $-2ax$ with the term $-6x$. To do this note that $-2a$ must be -6, so that $a = 3$.

Recall that

$$(x - a)^2 = x^2 - 2ax + a^2$$

With this value of a

$$(x - 3)^2 = x^2 - 6x + 9$$

However the right hand side is not yet the same as our original function $x^2 - 6x - 12$. To make it the same we subtract 9 from both sides:

$$(x - 3)^2 - 9 = x^2 - 6x + \not{9} - \not{9}$$

and then subtract 12 from each side:

$$(x - 3)^2 - 9 - 12 = x^2 - 6x - 12$$

So we have

$$\begin{aligned} x^2 - 6x - 12 &= (x - 3)^2 - 9 - 12 \\ &= (x - 3)^2 - 21 \end{aligned}$$

We have **completed the square**. We have written $x^2 - 6x - 12$ as a complete square $(x - 3)^2$ together with an additional term -21.

We know that because $f(x) = x^2 - 6x - 12$ has a positive x^2 term the graph will have a minimum value.

This will occur when $(x - 3)^2$ is zero. The minimum value of $f(x)$ will be -21 when $x = 3$.

Exercises

1. Complete the square for each of the following expressions

a)	$x^2 + 6x + 3$	b)	$x^2 - 10x - 6$	c)	$x^2 + 20x + 100$
d)	$x^2 - 5x + 2$	e)	$x^2 + x + 1$	f)	$x^2 - x + 1$

2. Find the minimum values of the following expressions

a)	$x^2 - x - 1$	b)	$x^2 + x - 1$	c)	$x^2 + 2x + 1$
d)	$x^2 - 8x + 5$	e)	$x^2 + \frac{1}{2}x + \frac{1}{2}$	f)	$x^2 - \frac{4}{5}x + \frac{1}{25}$

5. Example where the coefficient of x^2 is not 1

We now consider a more complicated example where the coefficient of x^2 is not 1.

Consider $f(x) = 2x^2 - 6x + 1$.

The first step is to take the 2 out as a common factor as follows:

$$2x^2 - 6x + 1 = 2\left(x^2 - 3x + \frac{1}{2}\right)$$

We now complete the square as before with the bracketed term.
Compare the two expressions:

$$x^2 - 2ax + a^2 \qquad \text{and} \qquad x^2 - 3x + \frac{1}{2}$$

Clearly the coefficients of x^2 in both expressions are the same - they match up.

We would like to match up the term $-2ax$ with the term $-3x$. To do this note that $2a$ must be 3, so that $a = \frac{3}{2}$.
Recall that

$$(x-a)^2 = x^2 - 2ax + a^2$$

With this value of a

$$\left(x - \frac{3}{2}\right)^2 = x^2 - 3x + \left(\frac{3}{2}\right)^2$$

However the right hand side is not yet the same as the original bracketed expression $\left(x^2 - 3x + \frac{1}{2}\right)$.

To make it the same we subtract $\left(\frac{3}{2}\right)^2$ from both sides

$$\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = x^2 - 3x + \cancel{\left(\frac{3}{2}\right)^2} - \cancel{\left(\frac{3}{2}\right)^2}$$

and then add $\frac{1}{2}$ to both sides:

$$\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{1}{2} = x^2 - 3x + \frac{1}{2}$$

So

$$\begin{aligned} x^2 - 3x + \frac{1}{2} &= \left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{1}{2} \\ &= \left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{1}{2} \\ &= \left(x - \frac{3}{2}\right)^2 - \frac{7}{4} \end{aligned}$$

So

$$2x^2 - 6x + 1 = 2\left(\left(x - \frac{3}{2}\right)^2 - \frac{7}{4}\right)$$

We have **completed the square** for the quadratic function $2x^2 - 6x + 1$.

The minimum value of the function $f(x)$ will be $2 \times \left(-\frac{7}{4}\right) = -\frac{7}{2}$ when $x = \frac{3}{2}$.

Example

In this Example we will consider a quadratic function for which the coefficient of x^2 is negative.

Consider the function $f(x) = 3 + 8x - 2x^2$. We operate in the same way as before, taking out the factor multiplying the x^2.

$$3 + 8x - 2x^2 = -2(x^2 - 4x - \frac{3}{2})$$

We now deal with just the bracketed term as before.
Compare the two expressions:

$$x^2 - 2ax + a^2 \qquad \text{and} \qquad x^2 - 4x - \frac{3}{2}$$

Clearly the coefficients of x^2 in both expressions are the same - they match up.

We would like to match up the term $-2ax$ with the term $-4x$. To do this note that $2a$ must be 4, so that $a = 2$. As before we note

$$(x - a)^2 = x^2 - 2ax + a^2$$

With $a = 2$

$$(x - 2)^2 = x^2 - 4x + 4$$

This is not yet the same as the original expression $x^2 - 4x - \frac{3}{2}$. To make it the same we can subtract 4 from each side:

$$(x - 2)^2 - 4 = x^2 - 4x + \not{4} - \not{4}$$

and subtract $\frac{3}{2}$ from each side:

$$(x - 2)^2 - 4 - \frac{3}{2} = x^2 - 4x - \frac{3}{2}$$

so that

$$\begin{aligned} x^2 - 4x - \frac{3}{2} &= (x - 2)^2 - 4 - \frac{3}{2} \\ &= (x - 2)^2 - \frac{11}{2} \end{aligned}$$

Then multiplying both sides by -2 in order to recover our original function

$$3 + 8x - 2x^2 = -2\left((x - 2)^2 - \frac{11}{2}\right)$$

We have **completed the square**.
In this case, when $x = 2$ the function will have its maximum value, and this will be 11.

Exercises

3. Complete the square for each of the following expressions

a) $2x^2 + 12x + 14$ b) $2x^2 + 12x + 13$ c) $3x^2 - 3x + 1$
d) $5x^2 + 4x + 3$ e) $10x^2 - 2x + 1$ f) $4x^2 - 10x - 6$

4. Find the minimum values of the following expressions

a) $2x^2 + 6x - 4$ b) $2x^2 + 8x - 1$ c) $5x^2 - 3x - 18$

5. Complete the square for each of the following expressions

a) $10 + 4x - x^2$ b) $12 - 9x - x^2$ c) $8 - 4x - 3x^2$
d) $7 + 6x - 2x^2$ e) $1 - 2x - 3x^2$ f) $1 + 2x - 3x^2$

6. Find the maximum values of the following expressions

a) $6 + 4x - x^2$ b) $8 - 6x - x^2$ c) $9 + 4x - 2x^2$

Answers

1. a) $(x+3)^2 - 6$ b) $(x-5)^2 - 31$ c) $(x+10)^2$
d) $\left(x - \frac{5}{2}\right)^2 - \frac{17}{4}$ e) $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ f) $\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$

2. a) $-\frac{5}{4}$ b) $-\frac{5}{4}$ c) 0 d) -11 e) $\frac{7}{16}$ f) $-\frac{3}{25}$

3. a) $2\left[(x+3)^2 - 2\right]$ b) $2\left[(x+3)^2 - \frac{5}{2}\right]$ c) $3\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{12}\right]$
d) $5\left[\left(x + \frac{2}{5}\right)^2 + \frac{11}{25}\right]$ e) $10\left[\left(x - \frac{1}{10}\right)^2 + \frac{9}{100}\right]$ f) $4\left[\left(x - \frac{5}{4}\right)^2 - \frac{49}{16}\right]$

4. a) $-\frac{17}{2}$ b) -9 c) $-\frac{369}{20}$

5. a) $-\left[(x-2)^2 - 14\right]$ b) $-\left[\left(x + \frac{9}{2}\right)^2 - \frac{129}{4}\right]$ c) $-3\left[\left(x + \frac{2}{3}\right)^2 - \frac{28}{9}\right]$
d) $-2\left[\left(x - \frac{3}{2}\right)^2 - \frac{23}{4}\right]$ e) $-3\left[\left(x + \frac{1}{3}\right)^2 - \frac{4}{9}\right]$ f) $-3\left[\left(x - \frac{1}{3}\right)^2 - \frac{4}{9}\right]$

6. a) 10 b) 17 c) 11