Solving Inequalities 3.5

Introduction

An inequality is an expression involving one of the symbols $\geq, \leq, >$ or $<$. This Section will first show how to manipulate inequalities correctly. Then algebraical and graphical methods of solving inequalities will be described.

Prerequisites

Before starting this Section you should ...

- be able to solve linear and quadratic equations

Learning Outcomes

On completion you should be able to ...

- re-arrange expressions involving inequalities
- solve linear and quadratic inequalities

1. The inequality symbols

Recall the definitions of the inequality symbols in Key Point 11:

Key Point 11

The symbols $>$, $<$, $\geq$, $\leq$ are called **inequalities**

$>$ means: 'is greater than', $\geq$ means: 'is greater than or equal to'

$<$ means: 'is less than', $\leq$ means: 'is less than or equal to'

So for example,

$$8 > 7 \qquad 9 \geq 2 \qquad -2 < 3 \qquad 7 \leq 7$$

A number line is often a helpful way of picturing inequalities. Given two numbers a and b, if $b > a$ then b will be to the right of a on the number line as shown in Figure 9.

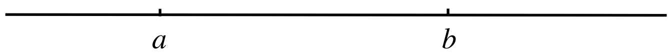

Figure 9: When $b > a$, b is to the right of a on the number line.

Note from Figure 10 that $-3 > -5$, $4 > -2$ and $8 > 5$.

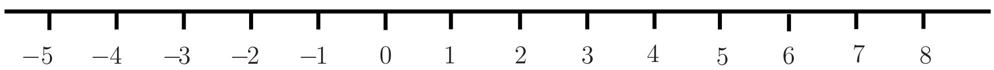

Figure 10

Inequalities can always be written in two ways. For example in English we can state that 8 is greater than 7, or equivalently, that 7 is less than 8. Mathematically we write $8 > 7$ or $7 < 8$. In general if $b > a$ then $a < b$. If $a < b$ then a will be to the left of b on the number line.

Example 33

Rewrite the inequality $-\frac{2}{5} < x$ using only the 'greater than' sign, $>$.

Solution

$-\frac{2}{5} < x$ can be written as $x > -\frac{2}{5}$

Example 34

Rewrite the inequality $5 > x$ using only the 'less than' sign, $<$.

Solution

$5 > x$ can be written as $x < 5$.

Sometimes two inequalities are combined into a single statement. Consider for example the statement $3 < x < 6$. This is a compact way of writing '$3 < x$ **and** $x < 6$'. Now $3 < x$ is equivalent to $x > 3$ and so $3 < x < 6$ means x is greater than 3 but less than 6.
Inequalities obey simple rules when used in conjunction with arithmetical operations:

Key Point 12

1. Adding or subtracting the same quantity from both sides of an inequality leaves the inequality symbol unchanged.
2. Multiplying or dividing both sides by a **positive** number leaves the inequality unchanged.
3. Multiplying or dividing both sides by a **negative** number reverses the inequality.

For example, since $8 > 5$, by adding k to both sides we can state

$$8 + k > 5 + k$$

for any value of k. For example (with $k = -3$) $8 - 3 > 5 - 3$. Further, by multiplying both sides of $8 > 5$ by k we can state $8k > 5k$ provided k is positive. However, $8k < 5k$ if k is negative.

We emphasise that the inequality sign is reversed when multiplying both sides by a negative number. A common mistake is to forget to reverse the inequality symbol. For example if $8 > 5$, multiplying both sides by -1 gives $-8 < -5$.

Task

Find the result of multiplying both sides of the inequality $-18 < 9$ by -3.

Your solution

Answer

$54 > -27$

The **modulus** or **magnitude** sign is sometimes used with inequalities. For example $|x| < 1$ represents the set of all numbers whose actual size, irrespective of sign, is less than 1. This means any value between -1 and 1. Thus

$|x| < 1$ means $-1 < x < 1$

Similarly $|x| > 4$ means all numbers whose size, irrespective of sign, is greater than 4. This means any value greater than 4 or less than -4. Thus

$|x| > 4$ means $x > 4$ or $x < -4$

In general, if k is a positive number:

Key Point 13

$|x| < k$ means $-k < x < k$

$|x| > k$ means $x > k$ or $x < -k$

Exercises

1. State which of the following statements are true and which are false.

(a) $4 > 9$, (b) $4 > 4$, (c) $4 \geq 4$, (d) $0.001 < 10^{-5}$, (e) $|-19| < 100$,
(f) $|-19| > -20$, (g) $0.001 \leq 10^{-3}$

In questions 2-9 rewrite each of the statements without using a modulus sign:

2. $|x| < 2$, 3. $|x| < 5$, 4. $|x| \leq 7.5$, 5. $|x-3| < 2$,
6. $|x-a| < 1$, 7. $|x| > 2$, 8. $|x| > 7.5$, 9. $|x| \geq 0$.

Answers

1. (a) F (b) F (c) T (d) F (e) T (f) T (g) T

2. $-2 < x < 2$ 3. $-5 < x < 5$ 4. $-7.5 \le x \le 7.5$

5. $-2 < x - 3 < 2$ 6. $-1 < x - a < 1$ 7. $x > 2$ or $x < -2$

8. $x > 7.5$ or $x < -7.5$ 9. $x \ge 0$ or $x \le 0$, in fact any x.

2. Solving linear inequalities algebraically

When we are asked to **solve an inequality**, the inequality will contain an unknown variable, say x. Solving means obtaining all values of x for which the inequality is true. In a **linear inequality** the unknown appears only to the first power, that is as x, and not as x^2, x^3, $x^{1/2}$ and so on.
Consider the following examples.

Example 35

Solve the inequality $4x + 3 > 0$.

Solution

$$\begin{aligned} 4x + 3 &> 0 \\ 4x &> -3, && \text{by subtracting 3 from both sides} \\ x &> -\frac{3}{4} && \text{by dividing both sides by 4.} \end{aligned}$$

Hence all values of x greater than $-\frac{3}{4}$ satisfy $4x + 3 > 0$.

Example 36

Solve the inequality $-3x - 7 \le 0$.

Solution

$$\begin{aligned} -3x - 7 &\le 0 \\ -3x &\le 7 && \text{by adding 7 to both sides} \\ x &\ge -\frac{7}{3} && \text{dividing both sides by } -3 \text{ and } \textbf{reversing the inequality} \end{aligned}$$

Hence all values of x greater than or equal to $-\frac{7}{3}$ satisfy $-3x - 7 \le 0$.

Task

Solve the inequality $17x + 2 < 4x + 1$.

This is done by making x the subject and obtain it on its own on the left-hand side.

Start by subtracting $4x$ from both sides to remove quantities involving x from the right:

Your solution

Answer

$13x + 2 < 1$

Now subtract 2 from both sides to remove the 2 on the left:

Your solution

Answer

$13x < -1$. Finally, the range of values of x are $x < -1/13$

Example 37

Solve the inequality $|5x - 2| < 4$ and depict the solution graphically.

Solution

$|5x - 2| < 4$ is equivalent to $-4 < 5x - 2 < 4$

We treat each part of the inequality separately:

$$\begin{aligned} -4 &< 5x - 2 & \\ -2 &< 5x & \text{by adding 2 to both sides} \\ -\frac{2}{5} &< x & \text{by dividing both sides by 5} \end{aligned}$$

So $x > -\frac{2}{5}$. Now consider the second part: $5x - 2 < 4$.

$$\begin{aligned} 5x - 2 &< 4 & \\ 5x &< 6 & \text{by adding 2 to both sides} \\ x &< \frac{6}{5} & \text{by dividing both sides by 5} \end{aligned}$$

So $x < \dfrac{6}{5}$.

Solution (contd.)

Putting both parts of the solution together we see that the inequality is satisfied when $-\frac{2}{5} < x < \frac{6}{5}$. This range of values is shown in Figure 11.

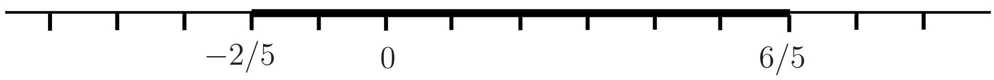

Figure 11: $|5x-2| < 4$ which is equivalent to $\frac{2}{5} < x < \frac{6}{5}$

Task

Solve the inequality $|1-2x| < 5$.

First of all rewrite the inequality without using the modulus sign:

Your solution

$|1-2x| < 5$ is equivalent to:

Answer

$-5 < 1-2x < 5$

Then treat each part separately. First of all consider $-5 < 1-2x$. Solve this:

Your solution

Answer

$x < 3$

The second part is $1-2x < 5$. Solve this.

Your solution

Answer

$x > -2$

Finally, give the solution as one statement:

Your solution

Answer

$-2 < x < 3$.

Exercises

In the following questions solve the given inequality algebraically.

1. $4x > 8$	2. $5x > 8$	3. $8x > 5$	4. $8x \leq 5$
5. $2x > 1$	6. $3x < -1$	7. $5x > 2$	8. $2x > 0$
9. $8x < 0$	10. $3x \geq 0$	11. $3x > 4$	12. $\frac{3}{4}x > 1$
13. $4x \leq -3$	14. $3x \leq -4$	15. $5x \geq 0$	16. $4x \leq 0$

17. $5x + 1 < 8$	18. $5x + 1 \leq 8$	19. $7x + 3 \geq 0$
20. $18x + 2 > 9$	21. $14x + 11 > 22$	22. $1 - 5x \leq 0$
23. $2 + 5x \geq 1$	24. $11 - 7x < 2$	25. $5 + 4x > 2x + 1$
26. $\lvert 7x - 3\rvert > 1$	27. $\lvert 2x + 1\rvert \geq 3$	28. $\lvert 5x\rvert < 1$
29. $\lvert 5x\rvert \leq 0$	30. $\lvert 1 - 5x\rvert > 2$	31. $\lvert 2 - 5x\rvert \geq 3$

Answers

1. $x > 2$	2. $x > 8/5$	3. $x > 5/8$	4. $x \leq 5/8$
5. $x > 1/2$	6. $x < -1/3$	7. $x > 2/5$	8. $x > 0$
9. $x < 0$	10. $x \geq 0$	11. $x > 4/3$	12. $x > 4/3$
13. $x \leq -3/4$	14. $x \leq -4/3$	15. $x \geq 0$	16. $x \leq 0$
17. $x < 7/5$	18. $x \leq 7/5$	19. $x \geq -3/7$	20. $x > 7/18$
21. $x > 11/14$	22. $x \geq 1/5$	23. $x \geq -1/5$	24. $x > 9/7$
25. $x > -2$	26. $x > 4/7$ or $x < 2/7$	27. $x \geq 1$ or $x \leq -2$	28. $-1/5 < x < 1/5$
29. $x = 0$	30. $x < -1/5$, $x > 3/5$	31. $x \leq -1/5$, $x \geq 1$	

3. Solving inequalities using graphs

Graphs can be used to help solve inequalities. This approach is particularly useful if the inequality is not linear as, in these cases solving the inequalities algebraically can often be very tricky. Graphics calculators or software can save a lot of time and effort here.

Example 38

Solve graphically the inequality $5x + 2 < 0$.

Solution

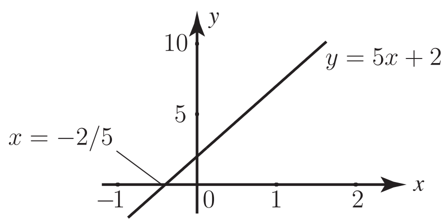

Figure 12: Graph of $y = 5x + 2$.

We consider the function $y = 5x + 2$ whose graph is shown in Figure 12. The values of x which make $5x + 2$ negative are those for which y is negative. We see directly from the graph that y is negative when $x < -\frac{2}{5}$.

Example 39

Find the range of values of x for which $x^2 - x - 6 < 0$.

Solution

We consider the graph of $y = x^2 - x - 6$ which is shown in Figure 13.

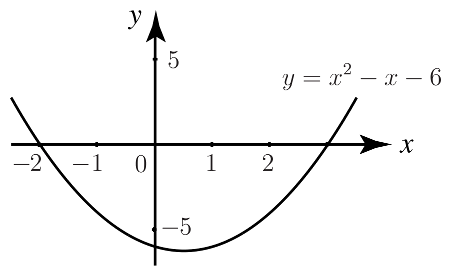

Figure 13: Graph of $y = x^2 - x - 6$

Note that the graph crosses the x axis when $x = -2$ and when $x = 3$, and $x^2 - x - 6$ will be negative when y is negative. Directly from the graph we see that y is negative when $-2 < x < 3$.

Task

Find the range of values of x for which $x^2 - x - 6 > 0$.

The graph of $y = x^2 - x - 6$ has been drawn in Figure 13. We require $y = x^2 - x - 6$ to be positive.

Use the graph to solve the problem:

Your solution

Answer

$x < -2$ or $x > 3$

Example 40

By plotting a graph of $y = 20x^4 - 4x^3 - 143x^2 + 46x + 165$ find the range of values of x for which

$$20x^4 - 4x^3 - 143x^2 + 46x + 165 < 0$$

Solution

A software package has been used to plot the graph which is shown in Figure 14. We see that y is negative when $-2.5 < x < -1$ and is also negative when $1.5 < x < 2.2$.

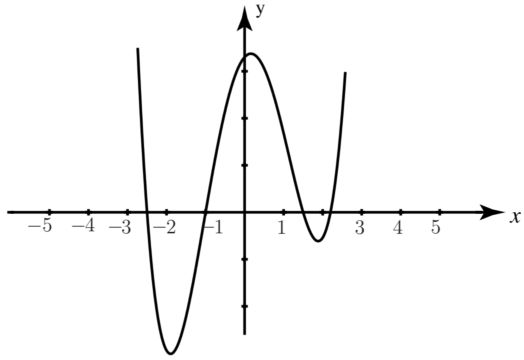

Figure 14: Graph of $y = 20x^4 - 4x^3 - 143x^2 + 46x + 165$

Exercises

In questions 1-5 solve the given inequality graphically:

1. $3x + 1 < 0$ 2. $2x - 7 < 0$ 3. $6x + 9 > 0$, 4. $5x - 3 > 0$ 5. $x^2 - x - 6 < 0$

Answers

1. $x < -1/3$ 2. $x < 7/2$, 3. $x > -3/2$ 4. $x > 3/5$ 5. $-2 < x < 3$