Decimals

In this unit we shall look at the meaning of decimals, and how they are related to fractions. We shall then look at rounding to given numbers of decimal places or significant figures. Finally we shall take a brief look at irrational numbers.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- multiply and divide a decimal by powers of ten;

- convert decimals into fractions, and vice versa;

- round a decimal number to a given number of decimal places, or to a given number of significant figures.

Contents

1. Introduction

A simple meaning of the word 'decimal' is 'connected with ten', and the decimal number system is a means of expressing any number from the very smallest to the very largest.

Let us look at how the system works. We have the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, they are our units, or 'ones'. To make all our other numbers we use these same digits, but in different places in our decimal system. For example, in the number 12, '2' is in the units place and stands for 2 units, and '1' is in the tens place and stands for one times ten, giving us $2 + 10 = 12$.

We can describe this by using a place value chart, and labelling the columns.

1000000	100000	10000	1000	100	10	1	•1/10	1/100	1/1000

The decimal point shows us where the fraction part of a number begins. To the left of the decimal point we have the whole number part. We have the units, the tens, hundreds, thousands, ten thousands, hundred thousands, millions and so on. You can see that, as we go from right to left, each column heading is ten times larger than its neighbour, so going in this direction we are multiplying by ten each time. On the other hand, if we go from left to right, each column heading is ten times smaller than its neighbour, so going in this direction we are dividing by ten.

To the right of the decimal point we have the fractional part of the number. If we continue going right, we continue dividing by ten. So we get tenths, hundredths, thousandths and so on.

We can put some numbers in the chart.

1000000	100000	10000	1000	100	10	1	•1/10	1/100	1/1000
					2	7			
				5	3	1			
					5	0			
			6	0	0	0			
				2	0	7			
	1	2	7	3	9	5			
						6	•3	9	2
						0	•5		
					1	2	•0	2	7

These numbers are 27, 531, 50, 6 000, 207, 127 395, 6.392, 0.5, and 12.027. In the chart, the numbers are aligned so that the units are vertically above or below one another, and similarly for the tens, the hundreds, and so on. For instance, 50 has a 5 in the tens column. But we also put a 0 in the units column, to show that there are no units. This 0 is called a 'placeholder', because it makes sure we realise that the 5 represents 5 tens, and not 5 units.

2. Multiplying and dividing by ten

Using our chart, we can see what happens when we multiply a number by ten. We shall work out 34×10.

1000000	100000	10000	1000	100	10	1	•1/10	1/100	1/1000
					3	4			
				↙	↙				
				3	4	0			

The 3 representing the tens becomes ten times bigger, and moves to the hundreds column. Similarly, the 4 representing the units becomes ten times bigger and moves to the tens column. Now the units column is empty, and since there are no tenths in the original number to make ten times bigger, we need to put a zero in the units column.

Next, we shall take 0.507 and multiply it by 100.

1000000	100000	10000	1000	100	10	1	•1/10	1/100	1/1000
						0	•5	0	7
					↙	↙	↙	↙	
				↙	↙	↙	↙		
				0	5	0	•7		

Each digit will now move two columns to the left. In this case the leading zero is no longer required, so the result is 50.7 .

With division, the process is similar. We shall use the chart to work out $127.5 \div 10$.

1000000	100000	10000	1000	100	10	1	•1/10	1/100	1/1000
				1	2	7	•5		
				↘	↘	↘	↘		
					1	2	•7	5	

The 1 in the hundreds column becomes ten times smaller, and moves to the tens column. The 2 in the tens column becomes ten times smaller, and moves to the units column. The 7 in the units column becomes ten times smaller, and moves to the tenths column. And finally, the 5 in the tenths column becomes ten times smaller, and moves to the hundredths column. So $127.5 \div 10 = 12.75$.

Next, we shall work out $2.3 \div 1000$. Here, we need to use an extra column on the right of our chart to include ten-thousandths, and so to save space we shall omit the column for millions. Notice that 1000 is $10 \times 10 \times 10 = 10^3$.

100000	10000	1000	100	10	1	•1/10	1/100	1/1000	1/10000
					2	•3			
						↘	↘		
							↘	↘	
								↘	↘
					0	•0	0	2	3

This time the 2 in the units column will move three places to the right, as we divide by 10 three times, moving from the units column to the thousandths column. Similarly, the 3 in the tenths column will move three places to the right, to the ten-thousandths column. Because there are no other digits, we need to place zeros in the tenths and hundredths columns to hold the place value of the other digits. In addition, we usually place a zero in the units column for clarity. So $2.3 \div 1000 = 0.0023$.

Next, we shall work out $7.1 \times 100\,000$.

100000	10000	1000	100	10	1	•1/10	1/100	1/1000	1/10000
					7	• 1			
				↙	↙				
			↙	↙					
		↙	↙						
	↙	↙							
↙	↙								
7	1	0	0	0	0				

This time the 7 in the units column will move five places to the left, as we multiply by 10 five times, moving from the units column to the hundred thousands column. Similarly, the 1 in the tenths column will move five places to the left, to the ten thousands column. Because there are no other digits, we need to place zeros in the thousands, hundreds, tens and units columns to hold the place value of the other digits. So $7.1 \times 100\,000 = 710\,000$.

Finally, we shall work out 7.1×10^{-3}. Now multiplying by 10^{-3} is the same as dividing by 10^3, that is, dividing by 1000.

100000	10000	1000	100	10	1	•1/10	1/100	1/1000	1/10000
					7	• 1			
						↘	↘		
							↘	↘	
								↘	↘
					0	• 0	0	7	1

So $7.1 \times 10^{-3} = 0.007\,1$.

Exercises

1. Carry out the following calculations:

(a) 16.21×1000 (b) 1.59×1000 (c) $0.37 \div 100$ (d) $27.321\,4 \div 10$.

3. Decimals and fractions

We shall now look at how decimals and fractions are related.

Decimals are, in fact, decimal fractions. If we put 0.2 on our place value chart, we see that the 2 is in the tenths column. So
$$0.2 \quad = \quad \frac{2}{10} \quad = \quad \frac{1}{5},$$
giving us a direct link between decimals and fractions. Similarly, if we look at 0.25, the 2 represents 2 tenths and the 5 represents 5 hundredths, so

$$0.25 \quad = \quad \frac{2}{10} + \frac{5}{100} \quad = \quad \frac{25}{100} \quad = \quad \frac{1}{4}$$

when written in its lowest form. And if we look at 134.526, this is

$$134 + \frac{5}{10} + \frac{2}{100} + \frac{6}{1000} \quad = \quad 134 + \frac{526}{1000} \quad = \quad 134 + \frac{263}{500} \quad = \quad 134\tfrac{263}{500}.$$

To turn fractions into decimal fractions we simply carry out the division. For example,

$$\frac{1}{2} \;\;=\;\; 1 \div 2 \;\;=\;\; 0.5$$

$$\frac{3}{4} \;\;=\;\; 3 \div 4 \;\;=\;\; 0.75$$

$$\frac{1}{3} \;\;=\;\; 1 \div 3 \;\;=\;\; 0.33333333\ldots \;\;=\;\; 0.\dot{3}$$

In this case, we can see that the decimal fraction of 1/3 is a non-terminating decimal, and we indicate that the 3 is repeated by writing it with a dot on top, as $\dot{3}$. So 1/3 as a fraction is a more precise notation than a decimal representation which stops after some number of places. The easiest way to convert a fraction to a decimal is to use a calculator to carry out the division, but you should be aware of the inaccuracy involved when the decimal does not terminate.

If you have a mixed fraction, for example $2\frac{5}{6}$, you turn it into a decimal by calculating $5 \div 6$ then adding the 2:

$$2\tfrac{5}{6} \;\;=\;\; (5 \div 6) + 2 \;\;=\;\; 0.8\dot{3} + 2 \;\;=\;\; 2.8\dot{3}\,.$$

Again in this example we have a non-terminating decimal, and to use $2\frac{5}{6}$ in decimal form we need to approximate, as we shall see in the next section.

Key Point

A decimal is a fraction written in decimal notation.

To find a decimal from a fraction, calculate the numerator divided by the denominator. If the decimal does not terminate and the final digit is repeated indefinitely, this is indicated by a dot placed over that digit, as in $\frac{1}{3} = 0.\dot{3}$. If a whole group of digits is repeated, this is indicated by dots placed over the first and last digits of the group, as in $\frac{1}{11} = 0.\dot{0}\dot{9}$.

Exercises

2. Write the following decimals as fractions:

  (a) 0.75   (b) 0.8   (c) 16.275   (d) 6.333333\ldots

3. Write the following fractions as decimals:

  (a) $\frac{6}{10}$ (b) $\frac{7}{11}$ (c) $2\frac{12}{48}$.

4. The accuracy of decimals

The accuracy of a number can be given using 'decimal places' or 'significant figures'. We shall look at decimal places first.

Suppose that your calculator gives you an answer of 32.7914, and you are asked to give the answer to one decimal place. This means that you write just 1 digit after the decimal point. So if we put a line after the first decimal place, we can clearly see the other decimal places that will be discarded.

$$32.7|914$$

So our first thought might be that the answer to one decimal place should be 32.7. But before writing our answer down, we need to inspect the decimal place that comes after the line we have drawn. If this digit is a 0, 1, 2, 3 or 4 then the answer required lies to the left of the line, as we had supposed. But if this digit is 5, 6, 7, 8 or 9, as in this example, then the answer is closer to the next tenth up, and the digit to the left of the line must be increased by 1. In this case, 32.79 is closer to 32.8 than it is to 32.7, so to one decimal place the answer is 32.8.

Here are some examples. Write 15.2172 to 3 decimal places (d.p.):

$$15.217|2 \text{ is } 15.217 \text{ to 3 d.p.}$$

Write 0.315 to 2 d.p.:

$$0.31|5 \text{ is } 0.32 \text{ to 2 d.p.}$$

When it comes to dealing with a 5, it is in fact exactly half way between and theoretically the number could be left alone or rounded up. However, mathematical convention is to round up.

Here are some more examples. Round 6.2549 to 1 d.p.:

$$6.2|549 \text{ is } 6.3 \text{ to 1 d.p.}$$

Now round it to 2 d.p.:

$$6.25|49 \text{ is } 6.25 \text{ to 2 d.p.}$$

Now round it to 3 d.p.:

$$6.254|9 \text{ is } 6.255 \text{ to 3 d.p.}$$

To round to significant figures (s.f.), we use the same principles but instead of starting at the decimal point and counting the number of places of decimals required, we start at the left-most digit that is not zero. For example, suppose we are asked to round 27.3721 to 3 s.f. Starting at the first non-zero digit from the left, we count 3 places and draw the line. Looking at the next digit after the line, we decide whether the previous digit stays the same or needs to be increased by 1. So:

$$27.3|721 \text{ is } 27.4 \text{ to 3 s.f.}$$

Round 0.005 214 to 1 s.f.: we start counting from the first non-zero digit on the left, and put the line after 1 digit. So:

$$0.005|214 \text{ is } 0.005 \text{ to 1 s.f.}$$

Write 27 413 200 to 2 s.f.: again start counting from the first non-zero digit on the left and put in the line. But remember, you need to fill in with zeros to ensure that the place values of the 2 and the 7 remain correct.

$$27|413\,200 \text{ is } 27\,000\,000 \text{ to 2 s.f.}$$

Exercises

4. Write these numbers to the given accuracy:

(a) 126.321 to 1 d.p.   (b) 0.579 138 to 1 d.p., and then 3 d.p.   (c) 0.525 to 2 dp

(d) 104 517 to 3 s.f.   (e) 0.000 897 to 1 s.f.   (f) 12.280 5 to 4 s.f.

5. Irrational numbers

Finally, we look briefly at the following question: what are irrational numbers?

Irrational numbers are numbers that cannot be written as proper fractions, that is as a/b where a and b are integers. In decimal fraction form, irrational numbers go on forever, and do not repeat in any pattern. For example

$$
\begin{aligned}
\pi &= 3.1415926\ldots \\
\sqrt{2} &= 1.414213\ldots \\
\sqrt{3} &= 1.73205\ldots \\
e &= 2.718281\ldots
\end{aligned}
$$

So if irrational numbers are in decimal form, they must be approximations to so many significant figures or decimal places.

It is worth noting that recurring decimals such as $0.333333\ldots$ are not irrational numbers, as they can be written as fractions, eg 1/3. For these numbers, there is a repeating pattern. An irrational number has no repeating pattern.

Answers

1.

(a) 16 210   (b) 1 590   (c) 0.003 7   (d) 2.732 14

2.

(a) $\frac{3}{4}$ (b) $\frac{4}{5}$ (c) $16\frac{11}{40}$ (d) $6\frac{1}{3}$

3.

(a) 0.6   (b) $0.6\dot{3}$ (c) 2.25

4.

(a) 126.3   (b) 0.6, 0.579   (c) 0.53   (d) 105 000   (e) 0.000 9   (f) 12.28