MHD WAVE MODES OF TWISTED MAGNETIC FLUX TUBE

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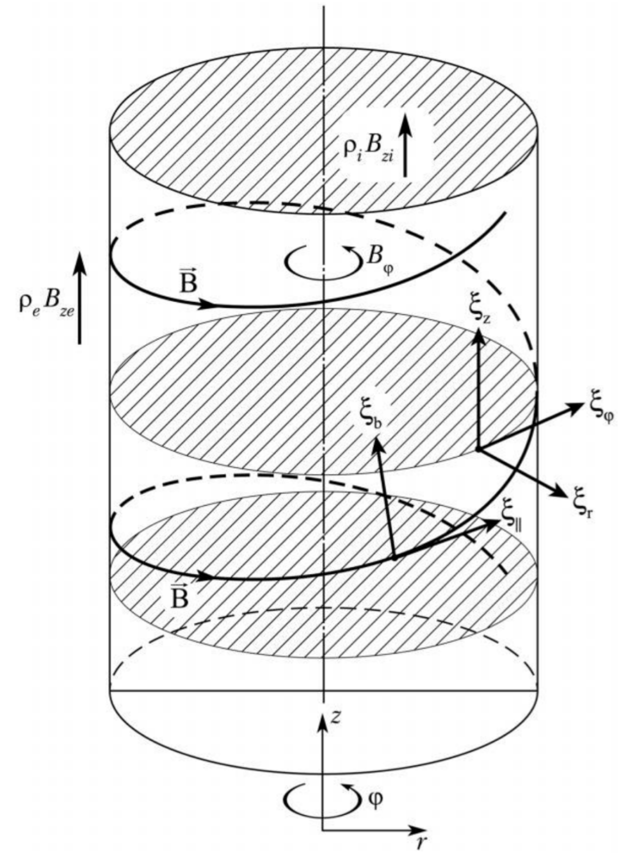
The governing wave equation for the linear radial component

An axially symmetric, vertical and magnetically twisted flux tube is a convenient model for analytical studies of various magnetohydrodynamic (MHD) perturbations

Assumptions

- MHD wave propagation in a magnetic flux tube with an internal twist only.
- To go beyond cold plasma equilibria conditions the vertical magnetic field inside and outside the flux tube are allowed to be different (similarly to Ryutov and Ryutova (1976), Bennett et al. (1999), Spruit, (1981, 1982); Edwin and Roberts, (1983), Goossens (2002), Erdélyi & Fedun (2007, 2010) +++.
- In the framework of ideal MHD, we assume incompressible linear perturbations and implement the thin tube approximation.
- We will focus on the analytical solutions related to modes with only m ≥ 1.



The geometry of the problem: a straight, vertical, uniformly twisted magnetic flux tube in an ambient magnetic field.

Both inside and outside the tube the equilibrium magnetic field is given by

$$\mathbf{B} = B_\varphi(r)\mathbf{e}_\varphi + B_z(r)\mathbf{e}_z$$

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By assuming the time dependence of all perturbed physical quantities as $exp\,(-i\omega t)$

these equations can be written as

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\rho\omega^2 \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi})$$

where

$$\mathbf{F}(\boldsymbol{\xi}) = -\nabla \delta p_1 + (\mathbf{B}\cdot\nabla)\,\delta\mathbf{B} + (\delta\mathbf{B}\cdot\nabla)\,\mathbf{B},$$
$$\delta p_1 = \delta p + \mathbf{B}\cdot\delta\mathbf{B} = -\gamma p \nabla\cdot\boldsymbol{\xi} - B^2\left(\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right),$$
$$\delta p = -\boldsymbol{\xi}\cdot\nabla p - \gamma p \nabla\cdot\boldsymbol{\xi},$$
$$\delta\mathbf{B} = \nabla\times[\boldsymbol{\xi}\times\mathbf{B}],$$
$$\nabla\cdot\delta\mathbf{B} = 0,$$
$$\boldsymbol{\kappa} = (\boldsymbol{\tau}\cdot\nabla)\,\boldsymbol{\tau},$$
$$\boldsymbol{\tau} = \mathbf{B}/B.$$

$$\mathbf{B}/\sqrt{4\pi} \to \mathbf{B}$$

δ corresponds to the perturbed quantities

ρ is the equilibrium plasma density

p is the equilibrium plasma pressure

$\boldsymbol{\xi} = \xi_r \mathbf{e}_r + \xi_\varphi \mathbf{e}_\varphi + \xi_z \mathbf{e}_z$ is the displacement vector

$\boldsymbol{\xi}_\perp = \boldsymbol{\xi} - \xi_\parallel \boldsymbol{\tau}$

γ is the adiabatic index

ω is the angular frequency

$\mathbf{B}$ is the equilibrium magnetic field

$\boldsymbol{\tau}$ is the normalised magnetic field

δp_1 is the perturbation of total plasma pressure

$\boldsymbol{\kappa}$ is the vector of curvature of magnetic field lines.

$$\boldsymbol{\kappa} = \left(\frac{\boldsymbol{B}}{B}\cdot\nabla\right)\frac{\boldsymbol{B}}{B}$$

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$$\rho\omega^2\xi_r-\frac{\mathrm{d}}{\mathrm{d}r}\delta p_1-\frac{2B_\varphi\delta B_\varphi}{r}+\mathrm{i}\left(\frac{m}{r}B_\varphi+k_zB_z\right)\delta B_r=0,$$
$$\rho\omega^2\xi_\varphi-\frac{\mathrm{i}m}{r}\delta p_1+\frac{\delta B_r}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rB_\varphi\right)+\mathrm{i}\left(\frac{m}{r}B_\varphi+k_zB_z\right)\delta B_\varphi=0,$$
$$\rho\omega^2\xi_z-\mathrm{i}k_z\delta p_1+\delta B_r\frac{\mathrm{d}}{\mathrm{d}r}B_z+\mathrm{i}\left(\frac{m}{r}B_\varphi+k_zB_z\right)\delta B_z=0.$$

We focus on all the modes with m ≥ 1, corresponding to non-axially symmetric oscillations which are the kink m = 1 and surface m > 1 modes

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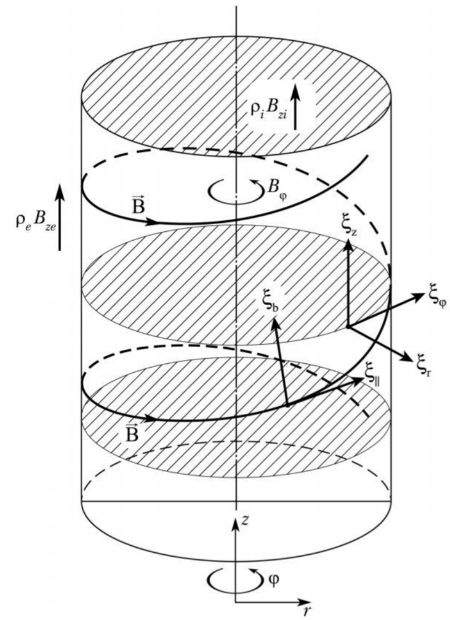
$$\rho\omega^2\xi_\varphi - \frac{\mathrm{i}m}{r}\delta p_1 + \frac{\delta B_r}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rB_\varphi\right) + \mathrm{i}\left(\frac{m}{r}B_\varphi + k_z B_z\right)\delta B_\varphi = 0,$$

$$\rho\omega^2\xi_z - \mathrm{i}k_z\delta p_1 + \delta B_r\frac{\mathrm{d}}{\mathrm{d}r}B_z + \mathrm{i}\left(\frac{m}{r}B_\varphi + k_z B_z\right)\delta B_z = 0.$$



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For convenience we changed φ and z components of $\boldsymbol{\xi}$ and $\boldsymbol{k}$ to The components directed along the bi-normal (subscript b) and along the magnetic field lines (subscrip $\parallel$):

$$\xi_b = \xi_\varphi\frac{B_z}{B} - \xi_z\frac{B_\varphi}{B}, \quad \boldsymbol{e}_b = \boldsymbol{e}_\varphi\frac{B_z}{B} - \boldsymbol{e}_z\frac{B_\varphi}{B},$$

$$\xi_\parallel = \xi_\varphi\frac{B_\varphi}{B} + \xi_z\frac{B_z}{B}, \quad \boldsymbol{e}_\parallel = \boldsymbol{e}_\varphi\frac{B_\varphi}{B} + \boldsymbol{e}_z\frac{B_z}{B},$$

$$k_b = \frac{m}{r}\frac{B_z}{B} - k_z\frac{B_\varphi}{B},$$

$$k_\parallel = \boldsymbol{k}\cdot\boldsymbol{e}_\parallel = \frac{m}{r}\frac{B_\varphi}{B} + k_z\frac{B_z}{B}.$$

The governing wave equation for the linear radial component

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\rho\left(\omega^2-\omega_\mathrm{A}^2\right)}{k_b^2+\chi^2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\xi_r)\right]+2r\xi_r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{B_\varphi^2}{r^2}\frac{\chi^2}{k_b^2+\chi^2}+\frac{B_\varphi B_z}{r^2}\frac{k_\|k_b}{k_b^2+\chi^2}\right]=\rho\left(\omega^2-\omega_\mathrm{A}^2\right)\xi_r+2\xi_r B_\varphi\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_\varphi}{r}\right)$$

$$-4\xi_r\frac{B_\varphi^2}{r^2\rho}\frac{\chi^2}{k_b^2+\chi^2}\frac{\left(k_\|B_z-k_\mathrm{b}B_\varphi\right)^2}{\left(\omega^2-\omega_\mathrm{A}^2\right)}$$

where

$$\omega_\mathrm{A}^2=k_\|^2c_\mathrm{A}^2,\quad \omega_\mathrm{S}^2=k_\|^2c_\mathrm{S}^2,\quad \omega_T^2=k_\|^2c_T^2,$$

$$c_\mathrm{A}^2=\frac{B^2}{\rho},\quad c_\mathrm{S}^2=\frac{\gamma p}{\rho},\quad c_T^2=\frac{c_\mathrm{S}^2}{1+\beta},\quad \beta=\frac{c_\mathrm{S}^2}{c_\mathrm{A}^2}$$

$$\chi^2=\frac{\left(\omega^2-\omega_\mathrm{A}^2\right)\left(\omega_\mathrm{S}^2-\omega^2\right)}{\left(c_\mathrm{S}^2+c_\mathrm{A}^2\right)\left(\omega^2-\omega_T^2\right)}$$

This equation is equivalent to the well known Hain-Lüst equation (Hain & Lüst 1958)

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where $\omega_A^2=k_\parallel^2c_A^2,\quad \omega_S^2=k_\parallel^2c_S^2,\quad \omega_T^2=k_\parallel^2c_T^2,$

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For incompressible perturbations :

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\rho\left(\omega^2-\omega_A^2\right)}{k_z^2+m^2/r^2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\xi_r)\right]+2r\xi_r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{BB_\varphi}{r^2}\frac{k_\parallel(m/r)}{k_z^2+m^2/r^2}\right]-\xi_r\left[\rho\left(\omega^2-\omega_A^2\right)+2B_\varphi\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_\varphi}{r}\right)-\frac{4\left(B_\varphi^2/r^2\right)}{k_z^2+m^2/r^2}\frac{k_z^2\omega_A^2}{\left(\omega^2-\omega_A^2\right)}\right]=0$$

The long wavelength approximation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\rho\left(\omega^2-\omega_{\mathrm{A}}^2\right)}{k_z^2+m^2/r^2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\xi_r)\right]+2r\xi_r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{BB_\varphi}{r^2}\frac{k_\parallel(m/r)}{k_z^2+m^2/r^2}\right]-\xi_r\left[\rho\left(\omega^2-\omega_{\mathrm{A}}^2\right)+2B_\varphi\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_\varphi}{r}\right)-\frac{4\left(B_\varphi^2/r^2\right)}{k_z^2+m^2/r^2}\frac{k_z^2\omega_{\mathrm{A}}^2}{\left(\omega^2-\omega_{\mathrm{A}}^2\right)}\right]=0$$

$$\varepsilon = k_z a \ll 1$$

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$$\varepsilon = k_z a \ll 1 \longrightarrow m^2+k_z^2r^2=m^2+\left(\frac{r}{a}\right)^2\varepsilon^2\approx m^2$$

The long wavelength approximation and boundary conditions

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\rho\left(\omega^2-\omega_{\mathrm{A}}^2\right)}{k_z^2+m^2/r^2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\xi_r\right)\right]+2r\xi_r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{BB_\varphi}{r^2}\frac{k_\parallel\left(m/r\right)}{k_z^2+m^2/r^2}\right]-\xi_r\left[\rho\left(\omega^2-\omega_{\mathrm{A}}^2\right)+2B_\varphi\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_\varphi}{r}\right)-\frac{4\left(B_\varphi^2/r^2\right)}{k_z^2+m^2/r^2}\frac{k_z^2\omega_{\mathrm{A}}^2}{\left(\omega^2-\omega_{\mathrm{A}}^2\right)}\right]=0$$

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$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\rho\omega^2-F^2\right)r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right]+\frac{\mathrm{d}F^2}{\mathrm{d}r}\frac{\phi}{r}-\left(\rho\omega^2-F^2\right)\frac{m^2\phi}{r^2}+4\frac{B_\varphi^2}{r^2}\frac{k_z^2F^2}{\left(\rho\omega^2-F^2\right)}\phi=0,$$

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Boundary conditions

$$\langle\phi\rangle=\phi(a+0)-\phi(a-0)=0.$$

$$\left\langle\delta p_1-\frac{B_\varphi^2}{r^2}\phi\right\rangle=0$$

General dispersion relation for m ≥ 1 modes

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\rho\omega^2 - F^2\right) r \frac{\mathrm{d}\phi}{\mathrm{d}r}\right] + \frac{\mathrm{d}F^2}{\mathrm{d}r}\frac{\phi}{r} - \left(\rho\omega^2 - F^2\right)\frac{m^2\phi}{r^2} + 4\frac{B_\varphi^2}{r^2}\frac{k_z^2 F^2}{(\rho\omega^2 - F^2)}\phi = 0,$$

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Wesson (1978)

study of stability of high-temperature plasma

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$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\rho\omega^2 - F^2\right) r^3 \frac{\mathrm{d}\xi_r}{\mathrm{d}r}\right] + \left(1 - m^2\right)\left(\rho\omega^2 - F^2\right)\xi_r = 0.$$

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For the specific kink mode value of m = 1

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\rho\omega^2 - F^2\right) r^3 \frac{\mathrm{d}\xi_r}{\mathrm{d}r}\right] = 0$$

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The physical solution of this equation for a trapped mode with shown above background magnetic field bounded at $r = 0$ and tending to $\xi_r = 0$ as as $r \to \infty$

$$\xi_r(r) = \begin{cases} \xi_a = \text{const.}, & r \le a \\ \xi_a\left(\frac{a}{r}\right)^2, & r > a. \end{cases}$$

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The physical solution of this equation for a trapped mode with shown above background magnetic field bounded at $r=0$ and tending to $\xi_r=0$ as as $r\to\infty$

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The fact that obtained solution is the same as for an untwisted tube is interesting analytical result since previously Ruderman (2007), for example, only demonstrated this for the particular internal background magnetic twist $B_\varphi \propto r$.

General dispersion relation for m ≥ 1 modes

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\rho\omega^2 - F^2\right) r^3 \frac{\mathrm{d}\xi_r}{\mathrm{d}r}\right] = 0$$

$$\boldsymbol{B} = \begin{cases} \left(0, B_\varphi(r), B_{\mathrm{zi}}\right), & r \le a \\ \left(0, 0, B_{\mathrm{ze}}\right) & r > a. \end{cases}$$

$$\xi_r(r) = \begin{cases} \xi_a = \mathrm{const.}, & r \le a \\ \xi_a \left(\frac{a}{r}\right)^2, & r > a. \end{cases}$$

Also, this result in an agreement with with the purely numerical study of Terradas & Goossens (2012) who solved the ideal linearised MHD equations in the zero-β regime

Terradas & Goossens (2012) found that the kink mode frequency in the long wavelength approximation was not affected by particular choice of a quadratic radial profile of B_φ

$$f(x) = \begin{cases} 0, & 0 \le x < p, \\ (x-p)(q-x), & p \le x \le q, \\ 0, & x > q, \end{cases}$$

General dispersion relation for m ≥ 1 modes

$$\left\langle \delta p_1 - \frac{B_\varphi^2}{r^2}\phi \right\rangle = 0$$

$$\xi_r(r) = \begin{cases} \xi_a = \text{const.}, & r \le a \\ \xi_a \left(\frac{a}{r}\right)^2, & r > a. \end{cases}$$

$$\delta p_1 = \frac{1}{m^2}\left[\frac{2m}{r}B_\varphi F\phi + \left(\rho\omega^2 - F^2\right) r \frac{\mathrm{d}\phi}{\mathrm{d}r}\right]$$

$$\phi = r\xi_r,\ F(r) = \frac{m}{r}B_\varphi(r) + k_z B_z$$

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⇓

$$(\rho_\mathrm{i} + \rho_\mathrm{e})\,\omega^2 = \left(F_\mathrm{i}(r)^2 + F_\mathrm{e}^2\right)\Big|_{r=a} - 2\frac{B_\varphi(r)F_\mathrm{i}(r)}{r}\Bigg|_{r=a} + \frac{B_\varphi^2(r)}{r}\Bigg|_{r=a}$$

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$$F_\mathrm{i}(r) = B_\varphi(r)/r + k_z B_{z\mathrm{i}},$$
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$$F_\mathrm{i}(r) = B_\varphi(r)/r + k_z B_{z\mathrm{i}},\quad F_\mathrm{e} = k_z B_{z\mathrm{e}},$$

$$\omega^2 = \frac{k_z^2}{(\rho_\mathrm{i} + \rho_\mathrm{e})}\left(B_{z\mathrm{i}}^2 + B_{z\mathrm{e}}^2\right)$$

is the same as the kink mode in the thin tube approximation without twist. Therefore, the frequency and radial displacement are unaffected by the choice of internal twist for the kink mode.

Internal background magnetic twist with $B_\varphi \propto r$

For fluting modes with m ≥ 2 we will assume that inside the tube the magnetic twist varies linearly and outside it is zero

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$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\rho\omega^2 - F^2\right) r^3 \frac{\mathrm{d}\xi_r}{\mathrm{d}r}\right] + \left(1 - m^2\right)\left(\rho\omega^2 - F^2\right)\xi_r = 0.$$

$$\boldsymbol{B} = \begin{cases} \left(0, B_\varphi(a)\, r/a, B_{z\mathrm{i}}\right), & r \le a \\ (0, 0, B_{z\mathrm{e}}) & r > a \end{cases}$$

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$$\xi_r = \begin{cases} \xi_a\left(\frac{r}{a}\right)^{m-1}, & r \le a \\ \xi_a\left(\frac{a}{r}\right)^{m+1}, & r > a. \end{cases}$$

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$$\omega^2 = \frac{1}{(\rho_i + \rho_e)}\left[k_z^2\left(B_{zi}^2 + B_{ze}^2\right) + m(m-1)\frac{B_\varphi^2(a)}{a^2} + \frac{2k_z}{a}(m-1)B_\varphi(a)B_{zi}\right].$$

Internal background magnetic twist with $B_\varphi \propto r$

For fluting modes with m ≥ 2 we will assume that inside the tube the magnetic twist varies linearly and outside it is zero

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\rho\omega^2 - F^2\right) r^3 \frac{\mathrm{d}\xi_r}{\mathrm{d}r}\right] + \left(1 - m^2\right)\left(\rho\omega^2 - F^2\right)\xi_r = 0.$$

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$$\left\langle\left(\rho\omega^2 - F^2\right) r \frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{2m}{r} B_\varphi F \phi - m^2 \left(\frac{B_\varphi}{r}\right)^2 \phi\right\rangle = 0.$$

$$\omega^2 = \frac{1}{(\rho_i + \rho_e)}\left[k_z^2\left(B_{zi}^2 + B_{ze}^2\right) + m(m-1)\frac{B_\varphi^2(a)}{a^2} + \frac{2k_z}{a}(m-1) B_\varphi(a) B_{zi}\right].$$

relation is invariant under the substitution $(m, k_z) \rightarrow (-m, -k_z)$, resulting in:

$$\omega^2 = \frac{1}{(\rho_i + \rho_e)}\left[k_z^2\left(B_{zi}^2 + B_{ze}^2\right) + m\left(m - \mathrm{sign}(m)\right)\frac{B_\varphi^2(a)}{a^2} + \frac{2k_z}{a}\left(m - \mathrm{sign}(m)\right) B_\varphi(a) B_{zi}\right].$$

Internal background magnetic twist with $B_\varphi \propto r$

$$\omega^2 = \frac{1}{(\rho_\mathrm{i} + \rho_\mathrm{e})} \left[k_z^2 \left(B_{z\mathrm{i}}^2 + B_{z\mathrm{e}}^2 \right) + m\,(m - \mathrm{sign}(m))\, \frac{B_\varphi^2(a)}{a^2} + \frac{2k_z}{a}\,(m - \mathrm{sign}(m))\, B_\varphi(a) B_{z\mathrm{i}} \right].$$

$$\Longleftrightarrow$$

$$\omega^2 = \frac{2B_0^2}{(\rho_\mathrm{i} + \rho_\mathrm{e})} \left\{ k_z^2 + \frac{A\,(m - \mathrm{sign}\,(m))}{2B_0^2}\,(Am + 2B_0 k_z) \right\}$$

Ruderman (2007)

$$B_\varphi(a)/a = A \text{ and } B_{z\mathrm{i}} = B_{z\mathrm{e}} = B_0$$

Internal background magnetic twist with $B_\varphi \propto r$

$$\omega^2 = \frac{1}{(\rho_i + \rho_e)}\left[k_z^2\left(B_{zi}^2 + B_{ze}^2\right) + m\,(m - \mathrm{sign}(m))\,\frac{B_\varphi^2(a)}{a^2} + \frac{2k_z}{a}\,(m - \mathrm{sign}(m))\,B_\varphi(a)B_{zi}\right].$$

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Ruderman (2007)

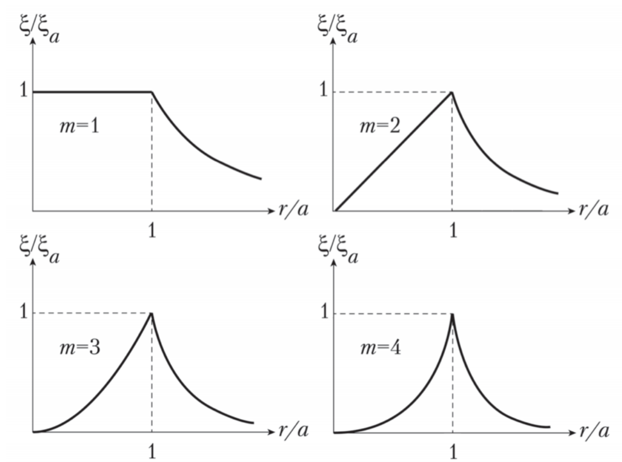
$$B_\varphi(a)/a = A \text{ and } B_{zi} = B_{ze} = B_0$$

The frequency of fluting modes (m ≥ 2) given by this equation, in contrast to kink modes (m = 1), has a minimum value when

$$k_z = -\frac{(m-1)}{a}\,\frac{B_\varphi(a)B_{zi}}{B_{zi}^2 + B_{ze}^2}$$

$$\Downarrow$$

$$\omega_{\min}^2 = \frac{B_\varphi^2(a)}{a^2}\,\frac{(m-1)}{(\rho_i + \rho_e)}\left[1 + \frac{(m-1)\,B_{ze}^2}{B_{zi}^2 + B_{ze}^2}\right]$$



For all m ≥ 2 modes the eigenfunction ξ_r has a form of power function and describes perturbations which are localised at the surface of the twisted magnetic flux tube.

Conclusions

- In the long wavelength limit, that both the frequency and radial velocity profile of the m = 1 kink mode are completely unaffected by the choice of internal background magnetic twist.
- Fluting modes with m ≥ 2 are sensitive to the particular radial profile of magnetic twist chosen.
- Due to background twist, a low frequency cut-off is introduced for fluting modes that is not present for kink modes. From an observational point of view, although magnetic twist does not affect the propagation of long wavelength kink modes, for fluting modes it will either work for or against the propagation, depending on the direction of wave travel relative to the sign of the background twist.